Realistic Roofs without Local Minimum Edges over a Rectilinear Polygon*

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Abstract

Computing all possible roofs over a given ground plan is a common task in automatically reconstructing a three dimensional building. In 1995, Aichholzer et al. proposed a definition of a *roof* over a simple polygon P in the xy-plane as a terrain over P whose faces are supported by planes containing edges of P and making a dihedral angle $\frac{\pi}{4}$ with the xy-plane. This definition, however, allows roofs with faces isolated from the boundary of P and local minimum edges inducing pools of rainwater. Very recently, Ahn et al. introduced "realistic roofs" over a rectilinear polygon with n vertices by imposing two additional constraints under which no isolated faces and no local minimum vertices are allowed. Their definition is, however, restricted and excludes a number of roofs with no local minimum edges. In this paper, we propose a new definition of realistic roofs over a rectilinear polygon that corresponds to the class of roofs without isolated faces and local minimum edges. We investigate the geometric and combinatorial properties of realistic roofs and show that the maximum possible number of distinct realistic roofs over a rectilinear n-gon is at most $1.3211^m\binom{m}{\lfloor \frac{m}{2} \rfloor}$, where $m = \frac{n-4}{2}$. We also present an algorithm that generates all combinatorial representations of realistic roofs.

1 Introduction

A common task in automatically reconstructing a three dimensional city model from its two dimensional map is to compute all the possible roofs over the ground plans of its buildings [4, 5, 11, 9, 10, 13]. For instance, Figure 1(a) shows a ground plan of a building in a perspective view, which is the union of two overlapping rectangles. The roof in Figure 1(b) can be constructed by building a roof over each rectangle and taking the upper envelope of the two roofs. The roof in Figure 1(c) can be constructed by shrinking the ground plan at a constant speed while moving it along vertically upward at a constant speed. Note that the vertical projection of the roof coincides with the the straight skeleton of the ground plan [2, 3].

For some applications, a correct or reasonable roof over a building is chosen from its set of possible roofs by considering some additional information such as its satellite images.

Aichholzer et al. [2] defined a *roof* over a simple (not necessarily rectilinear) polygon in the xy-plane as a terrain over the polygon such that the polygon boundary is contained in the terrain and each face of the terrain is supported by a plane containing at least one polygon edge and making a dihedral angle $\frac{\pi}{4}$ with the xy-plane. This definition, however, is not tight enough that it allows roofs with faces isolated from the boundary of the polygon (Figure 2(a)) and local minimum edges (Figure 2(b)) which are undesirable for some practical reasons – for example, a local minimum edge serves as a pool of rainwater, which can cause damage to the roof. Note that a pool of rainwater on a roof always contains a local minimum edge or vertex.

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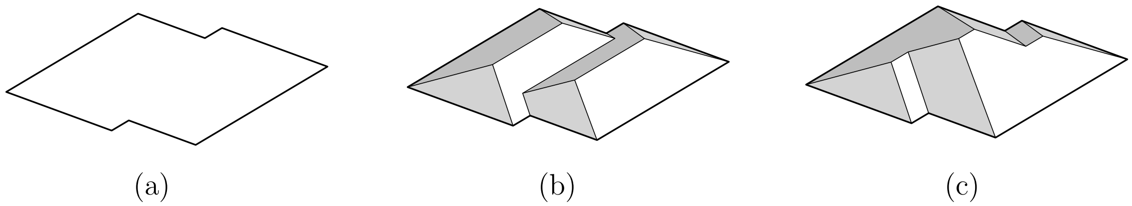

Figure 1: A rectilinear ground plan and two different roofs over the plan in a perspective view.

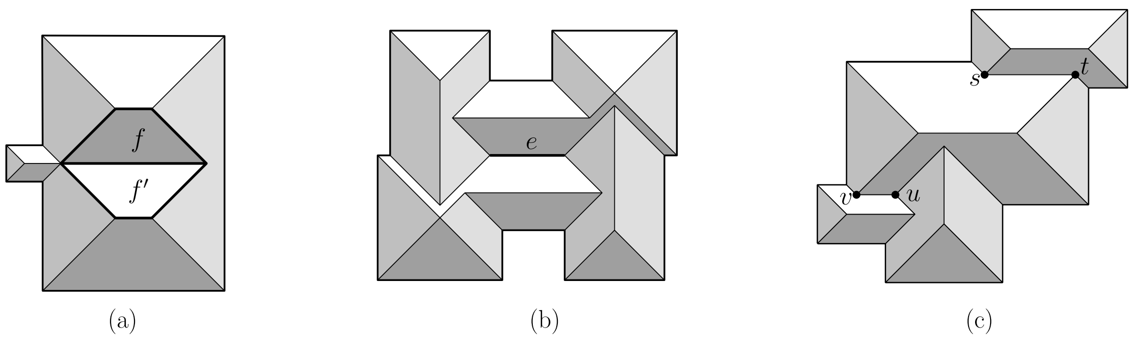

Figure 2: (a) A roof with isolated faces f and f'. (b) A roof with a local minimum edge e. (c) Not a realistic roof according to Definition 1; vertex u has no adjacent vertex that is lower than itself.

1.1 Related work

Brenner [5] designed an algorithm that computes all the possible roofs over a rectilinear polygon, but no polynomial bound on its running time is known. Recently, Ahn et al. [1] introduced "realistic roofs" over a rectilinear polygon P with n vertices by imposing two additional constraints to the definition of "roofs" by Aichholzer et al. [2] as follows.

Definition 1 ([1]) *A realistic roof over a rectilinear polygon P is a roof over P satisfying the following constraints.*

C1. Each face of the roof is incident to at least one edge of P.

C2. Each vertex of the roof is higher than at least one of its neighboring vertices.

They showed some geometric and combinatorial properties of realistic roofs, including a connection to the *straight skeleton* [2, 3, 7, 6, 8]. Consider a roof $R^*(P)$ over P constructed by a *shrinking process*, where all of the edges of P move inside, being parallel to themselves, with the same speed while moving upward along the z-axis with the same speed. Aichholzer et al. [2] showed that $R^*(P)$ is unique and its projection on the xy-plane is the straight skeleton of P. Ahn et al. [1] showed that $R^*(P)$ is the pointwise highest realistic roof over P. From the fact that $R^*(P)$ does not have a "valley", Ahn et al. [1] suggested a way of constructing another realistic roof over P different to $R^*(P)$ by adding a set of "compatible valleys" to $R^*(P)$. They showed that the number of realistic roofs lies between 1 and $\binom{m}{\lfloor \frac{m}{2} \rfloor}$ where $m = \frac{n-4}{2}$, and presented an output sensitive algorithm generating all combinatorial representations of realistic roofs over P in $O(1)$ amortized time per roof, after $O(n^4)$ preprocessing time.

1.2 Our results

Constraint *C1* in Definition 1 was introduced to exclude roofs with isolated faces and constraint *C2* was introduced to avoid pools of rainwater. However, *C2* is restricted and excludes a large number of roofs containing no local minimum edges. For example, the roof in Figure 2(c) is not realistic according to

Definition 1 though rainwater can be drained well along uv. Therefore, Definition 1 by Ahn et al. [1] does describe only a subset of "realistic" roofs.

We introduce a new definition of realistic roofs by replacing *C2* of Definition 1 with a relaxed one that excludes roofs with local minimum edges only.

Definition 2 *A realistic roof over a rectilinear polygon P is a roof over P satisfying the following constraints.*
C1. Each face of the roof is incident to at least one edge of P.
C2′. For each roof edge uv, u or v is higher than at least one of its neighboring vertices.

From now on, we use Definition 2 for realistic roofs unless stated explicitly. Our definition corresponds to the class of roofs without isolated faces, local minimum edges and local minimum vertices exactly.

Our main results are threefold:

1. We provide a new definition of "realistic roofs" that corresponds to the real-world roofs and investigate geometric properties of them.
2. We show that the maximum possible number of realistic roofs over a rectilinear n-gon is at most $1.3211^m\binom{m}{\lfloor \frac{m}{2} \rfloor}$, where $m = \frac{n-4}{2}$.
3. We present an algorithm that generates all combinatorial representations of realistic roofs over a rectilinear n-gon. Precisely, it generates a roof whose vertices are all open, that is, every vertex is higher than at least one of its neighboring vertices in $O(1)$ time after $O(n^4)$ preprocessing time [1]. For each such roof R, it generates $O(1.3211^m)$ realistic roofs in time $O(m1.3211^m)$ by adding edges on R.

2 Preliminary

For a point p in $\mathbb{R}^3$, we denote by $x(p), y(p)$, and $z(p)$ the x-, y-, and z-coordinate of p, respectively. We denote by $\overline{p}$ the orthogonal projection of p onto the xy-plane. A line through $\overline{p}$ parallel to the x-axis, and another line through $\overline{p}$ parallel to the y-axis together divide the xy-plane into four regions, called *quadrants* of $\overline{p}$, each bounded by two half-lines. For a point q in a roof R, let $D(q)$ be the axis-parallel square centered at $\overline{q}$ with side length $2z(q)$.

We denote by ∂P the boundary of P and by edge(f) the edge of ∂P incident to a face f of a roof.

Lemma 1 ([1]) *Let R be a roof over a rectilinear polygon P. The followings hold.*
(a) For any point $p \in R$, $z(p)$ is at most the L_∞ distance from $\overline{p}$ to its closest point in ∂P. Therefore, we have $D(p) \subseteq P$.
(b) For each edge e of P, there exists a unique face f of R incident to e.
(c) Every face f of R is monotone with respect to the line containing edge(f).

Consider the boundary ∂f of f. According to property (c) of Lemma 1, ∂f consists of exactly two chains monotone with respect to the line containing edge(f).

An edge e of a realistic roof R over P is *convex* if the two faces incident to e make a dihedral angle below R less than π, and *reflex* otherwise. A convex edge is called *ridge* if it is parallel to the xy-plane. A reflex edge is called a *valley* if it is parallel to the xy-plane.

3 Valleys of a Realistic Roof

In this section, we investigate local structures of realistic roofs. Ahn et al. [1] showed five different configurations of end vertices that a ridge can have under Definition 1. They also showed that vertices which are not incident to a valley or a ridge are degenerated forms of valleys or ridges. Since replacing constraint *C2* with *C2′* does not affect ridges, we care about only valleys.

We define three types of valleys and describe their structures that a realistic roof can have. We call a vertex of a roof *open* if it is higher than at least one of its neighboring vertices connected by roof edges, and *closed* otherwise. We call a valley *open* if both end vertices are open, *half-open* if one end vertex is open and the other is closed, and *closed* if both end vertices are closed. For instance, the valley uv in Figure 2(c) has an open end vertex v and a closed end vertex u, and therefore it is half-open.

By Definition 2, a realistic roof can contain open and half-open valleys but it does not contain closed valleys. Ahn et al. [1] showed that each open valley always has the same structure as *st* in Figure 2(c). More specifically, they first showed that there are only 5 possible configurations near an end vertex of a valley which satisfy the roof constraints such as the monotonicity of a roof, and the slope and orientations of faces as illustrated in Figure 3. Then they showed that an open valley must have both end vertices of configuration (v1) only and oriented symmetrically along the valley. Otherwise, an end vertex of the valley becomes a local minimum or a face f incident to the valley is not monotone with respect to the line containing edge(f) contradicting Lemma 1(c). They also observe that each end vertex of an open valley is connected to a reflex vertex of P by a reflex edge. We call such a reflex vertex a *foothold* of the open valley. Note that two footholds a and a' of an open valley uv are opposite corners of $B_{aa'}$ and $B_{aa'} \setminus \{a, a'\}$ is contained in the interior of P, and uv coincides with the ridge of $R^*(B_{aa'})$, where $B_{aa'}$ denote the smallest axis-aligned rectangle containing a and a' .

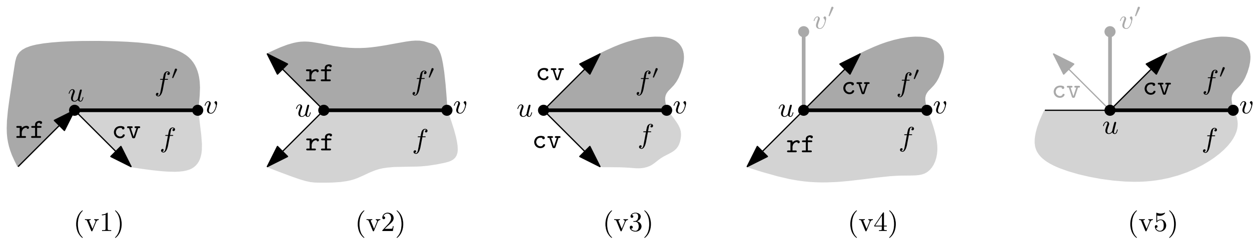

Figure 3: Five possible configurations around a vertex u of a valley uv shown by Ahn et al. [1], where `rf` denotes a reflex edge and `cv` denotes a convex edge. Each convex or reflex edge is oriented from the endpoint with smaller z-coordinate to the other one with larger z-coordinate.

In the following we investigate the structure of a half-open valley that a realistic roof can have. It is not difficult to see that the open end vertex is always of configuration (v1); and any end vertex of the other configurations cannot have a lower neighboring vertex. We will show that every closed end vertex of a valley is always of configuration (v2). For this, we need a few technical lemmas.

Lemma 2 *Let uv be a valley and uv' be a convex edge incident to u. Also, let ℓ be the line in the xy-plane passing through $\overline{v}$ and orthogonal to $\overline{uv}$. Then the face f incident to both uv and uv' has* edge(f) *in the half-plane of ℓ in the xy-plane not containing $\overline{u}$.*

Proof. Figure 3 shows all possible configurations that an end vertex u of a valley uv has. Since uv' is convex, v' is strictly higher than u and $\overline{uv'}$ makes an angle 45° with $\overline{uv}$ in all cases. Then the lemma follows from the monotonicity property (c) of Lemma 1. □

Imagine that a face f is incident to a valley uv and two convex edges one of which is incident to u and the other to v. This is only possible when both convex edges lie in the same side of the plane containing uv and parallel to the z-axis, because of the monotonicity of a roof, and the slope and orientations of faces. Since both convex edges make an angle 45° with uv in their projection on the xy-plane, f cannot have a ground edge by Lemma 2, that is, f is *isolated*.

Lemma 3 *Let uv be a half-open valley of a realistic roof where u is closed and v is open. Then v is of configuration* (v1) *and u is of configuration* (v2)*.*

Proof. If u is of configuration (v3), then one of two faces incident to uv becomes isolated by Lemma 2. If u is of configuration (v5), then there always is another valley uv' that is orthogonal to uv and has a closed corner at u of configuration (v3) as shown in Figure 3. Therefore one of faces incident to uv' is isolated.

Assume now that u is of configuration (v4). Then there always is another valley uv' orthogonal to uv. Therefore, we need to check two connected valleys uv and uv' simultaneously. Figure 4 illustrates all possible combinations of these two valleys. For cases (a) and (b), there is an isolated face incident to uv or uv'. For case (c), let f and f' be the faces incident to uv and uv', respectively, sharing the reflex edge incident to u as shown in Figure 4(c). By Lemma 2, edge(f) must lie in the top right quadrant of $\overline{u}$ and edge(f') must lie in the bottom left quadrant of $\overline{u}$ in the xy-plane. This is, however, not possible unless f or f' violates the monotonicity property (c) of Lemma 1.

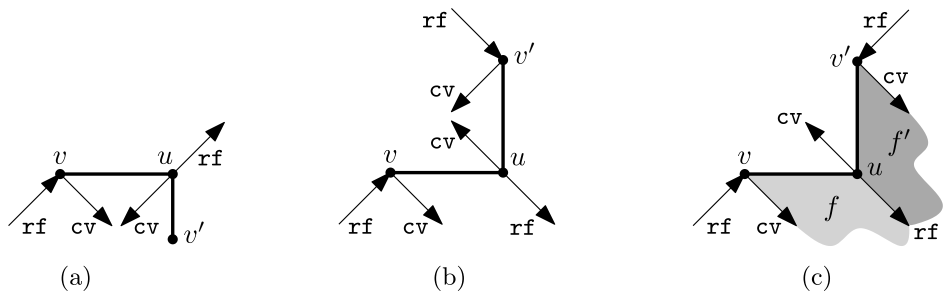

Figure 4: Three possible combinations around a (v4) type vertex.

The only remaining closed end vertex is of configuration (v2). Figure 5 shows a half-open valley uv with u of configuration (v1) and v of configuration (v2). □

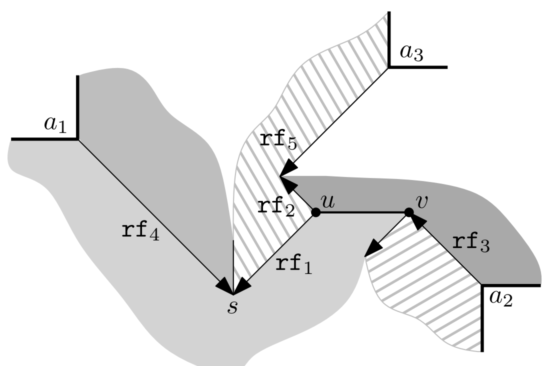

Figure 5: A half-open valley uv must be connected to three reflex vertices a_1, a_2 and a_3 of P via five reflex edges. We call the vertex s which is incident to $\mathtt{rf}_1$ and $\mathtt{rf}_4$ the *peak point* of uv.

Now we are ready to describe the structure of a half-open valley. In the following, we show that a half-open valley always has the same structure on a realistic roof as in Figure 5. Specifically, a half-open valley uv is associated with five reflex edges of the roof and three reflex vertices of P which have mutually different orientations. We call the three reflex vertices of P that induce a half-open valley the *footholds* of the valley.

Open vertex v to foothold a_2 Suppose that $\mathtt{rf}_3$ in Figure 5 is not connected to a reflex vertex of P. Then $\mathtt{rf}_3$ must be incident to another half-open valley $u'v'$, because a closed vertex of configuration (v2) is the only roof vertex that can have such a reflex edge. By Lemma 3, there are four possible cases and they are illustrated in Figure 6.

In case (a), face f_1 is isolated by the monotonicity property (c) of Lemma 1. In case (b), by the monotonicity of f_1, edge(f_1) must lie in the top left quadrant of $\overline{u}$ in the xy-plane. This implies that edge(f_2) must lie in the top right quadrant of $\overline{u}$, and edge(f_3) must lie in the bottom right quadrant of $\overline{u'}$ in

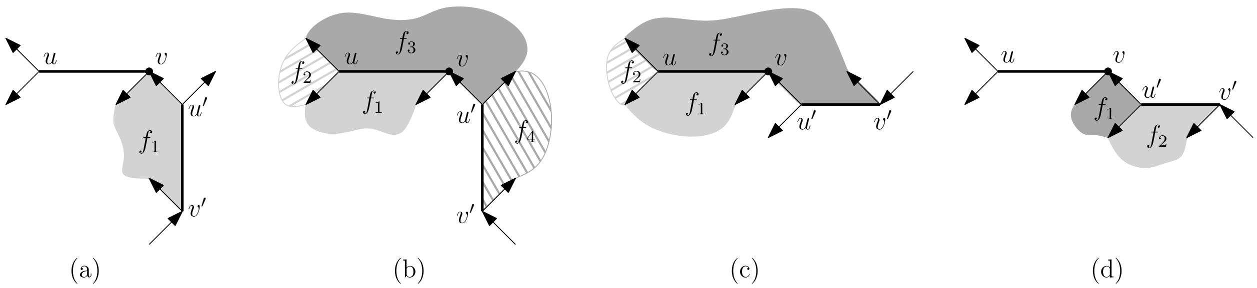

Figure 6: Four possible cases of two half-open valleys, uv and $u'v'$, connected by reflex edge $u'v$.

the xy-plane. However, by the monotonicity of f_4, edge(f_4) must lie in the top left quadrant of $\overline{u'}$, and this is not possible unless f_3 or f_4 violates the monotonicity property (c) of Lemma 1. In case (c), edge(f_1) must lie in the top left quadrant and edge(f_3) must lie in the bottom left quadrant of $\overline{u}$ in the xy-plane. Then f_1 or f_3 violates the monotonicity property. In case (d), edge(f_1) must lie in the bottom right quadrant and edge(f_2) must lie in the top left quadrant of $\overline{u'}$ in the xy-plane. This is, however, not possible unless f_1 or f_2 violates the monotonicity property. Therefore, v must be connected to a reflex vertex a_2 of P via $\texttt{rf}_3$.

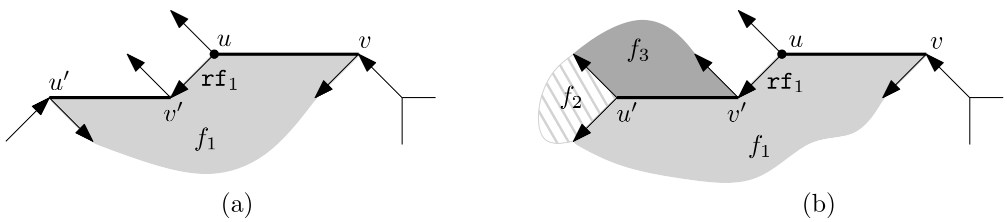

Figure 7: When $\texttt{rf}_1$ is connected to either (a) an open valley $u'v'$ or (b) a half-open valley $u'v'$.

Closed vertex u to footholds a_1 and a_3 We show that u is connected to foothold a_1 via two reflex edges $\texttt{rf}_1$ and $\texttt{rf}_4$. Note that the end vertex of $\texttt{rf}_1$ other than u is an end vertex (of configuration (v1)) of a valley or a ridge.

When $\texttt{rf}_1$ is connected to an open valley $u'v'$, both uv and $u'v'$ are incident to a face f_1, which is isolated. See Figure 7(a). If $u'v'$ is a half-open valley, then one of two faces incident to $u'v'$ violates the monotonicity (c) of Lemma 1. See Figure 7(b).

When $\texttt{rf}_1$ is connected to a ridge, there is another reflex edge $\texttt{rf}_4$ incident to the ridge. Suppose that $\texttt{rf}_4$ is not connected to a reflex vertex of P. Then $\texttt{rf}_4$ must be incident to another half-open valley $u'v'$. Figure 8 shows all four possible cases, but none of them can be constructed in a realistic roof: either a face is isolated (cases (a) and (c)) or at least one face violates the monotonicity (c) of Lemma 1 (cases (b) and (d)). Therefore, u must be connected to a reflex vertex a_1 of P via two reflex edges $\texttt{rf}_1$ and $\texttt{rf}_4$.

In a similar way, we can show how u is connected to foothold a_3 via two reflex edges $\texttt{rf}_2$ and $\texttt{rf}_5$.

Lemma 4 *Let uv be a half-open valley where u is closed and v is open. Then uv is associated with three reflex vertices of P that have mutually different orientations as shown in Figure 5.*

4 Realistic Roofs with Half-Open Valleys

From Lemma 4, we know that a half-open valley is associated with three reflex vertices that have mutually different orientations. In the following we investigate a condition under which three reflex vertices a_1, a_2, and a_3 with mutually different orientations can *induce* a half-open valley.

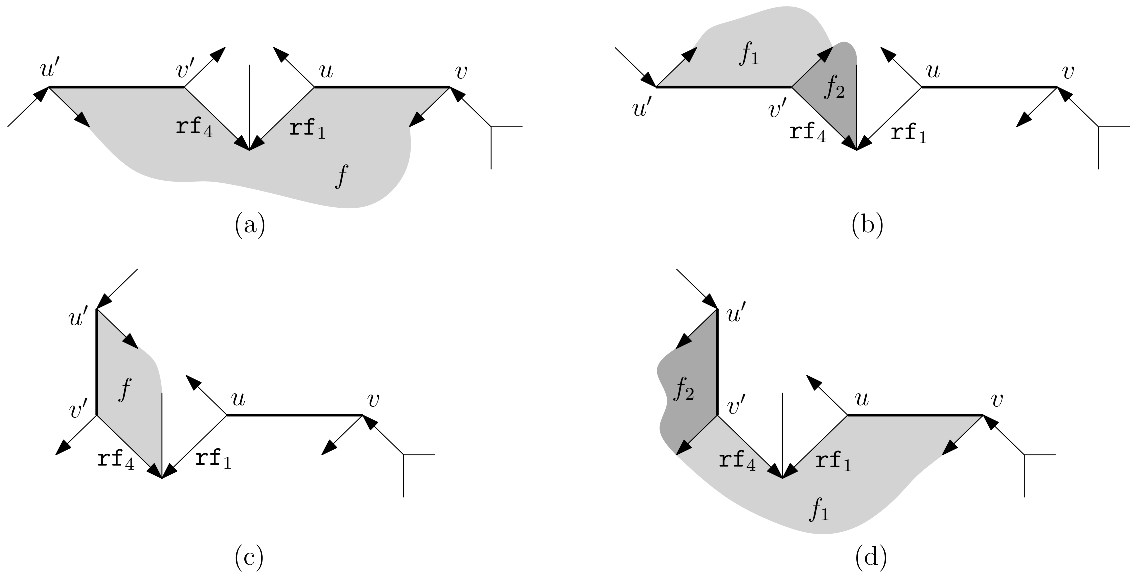

Figure 8: When $\texttt{rf}_4$ is connected to another half-open valley $u'v'$.

Let $d_x(i,j) := x(a_i) - x(a_j)$ and $d_y(i,j) := y(a_i) - y(a_j)$. Without loss of generality, we assume that these three vertices are oriented and placed as in Figure 5. That is, we have $d_x(3,1), d_x(2,3), d_y(1,2), d_y(3,1) > 0$. We define two squares and one rectangle in the xy-plane to determine whether these three reflex vertices form a half-open valley. Let r_1 be the square with a_1 on its top left corner and side length $d_x(3,1)$. Let r_2 be the rectangle with a_2 on its bottom right corner with height $d_y(1,2)$ and width $d_y(1,2) + d_x(2,3)$. Finally, let r_3 be the square with a_3 on its top right corner and side length $d_y(3,2)$. Note that these three rectangles overlap each other and have a nonempty common intersection.

We define three rectilinear subpolygons of P along r_1, r_2, and r_3 as follows. Let $P' := P \setminus (r_1 \cup r_2 \cup r_3)$. Let P_1 denote the union of $r_1 \cup r_2$ and the components of P' incident to the portion of ∂P from a_1 to a_2 in a counterclockwise direction (Figure 9(a)). Let P_2 denote the union of $r_2 \cup r_3$ and the components of P' incident to the portion of ∂P from a_2 to a_3 in a counterclockwise direction (Figure 9(b)). Let P_3 denote the union of $r_1 \cup r_3$ and the components of P' incident to the portion of ∂P from a_3 to a_1 in a counterclockwise direction (Figure 9(c)).

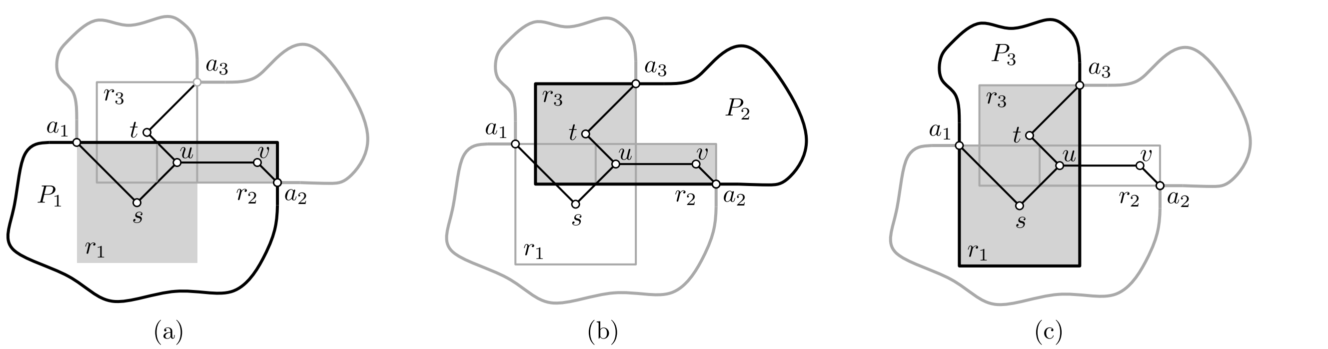

Figure 9: Dividing P into three rectilinear subpolygons, P_1, P_2 and P_3, *along* a half-open valley uv.

Lemma 5 *There is a realistic roof with a half-open valley induced by reflex vertices a_1, a_2 and a_3 of P if and only if $(r_i \setminus a_i) \cap \partial P = \emptyset$, for all $i \in \{1, 2, 3\}$.*

Proof. Let uv be the half-open valley of a realistic roof R induced by a_1, a_2 and a_3. We know that uv is connected to a_1, a_2 and a_3 via five reflex edges as shown in Figure 9. Note that $r_1 = D(s), r_3 = D(t)$, and $r_2 = \bigcup_{p \in uv} D(p)$. Therefore, $r_i \subseteq P$ for all $i \in \{1, 2, 3\}$. Let S_ε denote the set of points on R in the ε-neighborhood of s for small $\varepsilon > 0$. By property (a) of Lemma 1, we have $D(p) \subseteq P$ for every $p \in S_\varepsilon$. Since s is an end vertex of a ridge and it is incident to two reflex edges, $\bigcup_{p \in S_\varepsilon} D(p)$ contains ∂r_1 in its interior, except a_1 and the top right corner of r_1. The top right corner of r_1 coincides with the top right corner of $D(u)$, and there is a point q on R near u such that $D(q)$ contains the top right corner of r_1 in its interior. By using a similar argument, we can show that $(r_3 \setminus a_3) \cap \partial P = \emptyset$. For r_2, let U_ε denote the set of points on R in the ε-neighborhood of uv for small $\varepsilon > 0$. Since uv is a half-open valley, $\bigcup_{p \in U_\varepsilon} D(p)$ contains ∂r_2 in its interior, except a_2.

Now assume that $(r_i \setminus a_i) \cap \partial P = \emptyset$ for all $i \in \{1, 2, 3\}$. We will show that the upper envelope of $R^*(P_1) \cup R^*(P_2) \cup R^*(P_3)$ forms a realistic roof R over P which contains the unique half-open valley uv induced by a_1, a_2 and a_3. Since P_1 and P_2 both contain r_2, $R^*(P_1)$ and $R^*(P_2)$ intersect along a_2v and uv. Likewise, P_2 and P_3 both contain r_3, so $R^*(P_2)$ and $R^*(P_3)$ intersect along a_3t and ut. Finally, P_1 and P_3 both contain r_1, so $R^*(P_1)$ and $R^*(P_3)$ intersect along a_1s and us. Therefore uv and its five associated reflex edges appears on R.

It remains to show that every face f on the upper envelope of $R^*(P_1) \cup R^*(P_2) \cup R^*(P_3)$ is not isolated and monotone along the line containing edge(f). Since all faces in $R^*(P_i)$, for all $i \in \{1, 2, 3\}$ satisfy the condition, it suffices to consider only faces incident to uv and its five associated reflex edges.

Consider the face f_1 that is incident to $uv, \texttt{rf}_1$ and $\texttt{rf}_4$. Since r_1 touches ∂P only at a_1, there exists a rectangle $r'_1 \subseteq P_1$ that contains r_1 and whose boundary contains the top side of r_1 only. Since r_2 touches ∂P only at a_2, there exists a rectangle $r'_2 \subseteq P_1$ that contains r_2 and whose boundary contains the top and right sides of r_2 only. See Figure 10 (a). Then f_1 has the horizontal edge of P incident to a_1 as edge(f_1).

Likewise, there exist rectangles $r'_2, r'_3 \subseteq P_2$ such that $r_2 \subset r'_2$ and $r_3 \subset r'_3$, and therefore face f_2 incident to $uv, \texttt{rf}_2$ and $\texttt{rf}_3$ has the horizontal edge of P incident to a_2 as edge(f_2). See Figure 10 (b).

Finally, there exist rectangles $r'_1, r'_3 \subseteq P_3$ such that $r_1 \subset r'_1$ and $r_3 \subset r'_3$, and therefore face f_3 incident to $\texttt{rf}_1, \texttt{rf}_2$ and $\texttt{rf}_5$ has the vertical edge of P incident to a_3 as edge(f_3). See Figure 10 (c).

Clearly, face f_i is monotone with respect to edge(f_i) for all $i \in \{1, 2, 3\}$. ◻

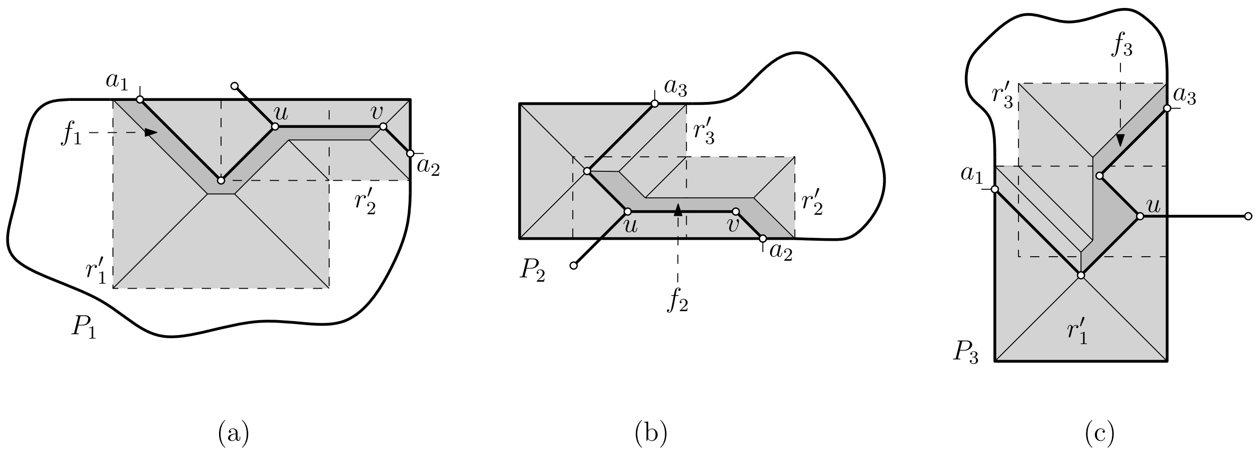

Figure 10: A half-open valley uv can be constructed by taking upper envelope of $R^*(P_1) \cup R^*(P_2) \cup R^*(P_3)$. (a) Face f_1 has the horizontal edge incident to a_1 as edge(f_1), (b) face f_2 has the horizontal edge incident to a_2 as edge(f_2), and (c) face f_3 has the vertical edge incident to a_3 as edge(f_3).

Assume that three reflex vertices of a candidate triple are oriented and placed as depicted in Figure 5. If three reflex vertices a_1, a_2 and a_3 satisfy the conditions in Lemma 5, we call (a_1, a_2, a_3) a *candidate triple* of footholds for uv, and $\bigcup_{i \in \{1,2,3\}} r_i$ the *free space* of uv.

Compatibility Given candidate pairs and triples of footholds for open and half-open valleys, respectively, we need to check whether there is a realistic roof that contains these valleys. In some cases, there is no realistic roof that contains two given valleys because of the geometric constraints of realistic roofs. We say a pair of valleys are *compatible* if there is a realistic roof that contains them.

We start with a lemma which states the compatibility between two open valleys.

Lemma 6 ([1]) *Let (a_1, a_2) and (a'_1, a'_2) be two candidate pairs of footholds for open valleys uv and $u'v'$, respectively. (a_1, a_2) and (a'_1, a'_2) are compatible if and only if $\overline{C}_{a_1a_2} \cap \overline{C}_{a'_1a'_2} = \emptyset$, where $\overline{C}_{a_1a_2} := a_1\overline{u} \cup \overline{uv} \cup \overline{v}a_2$ and $\overline{C}_{a'_1a'_2} := a'_1\overline{u'} \cup \overline{u'v'} \cup \overline{v}a'_2$.*

There are two cases to consider: compatibility between two half-open valleys, and compatibility between an open valley and a half-open valley.

Lemma 7 *Let (a_1, a_2, a_3) and (a'_1, a'_2, a'_3) be candidate triples of footholds for two half-open valleys uv and $u'v'$. Two half-open valleys uv and $u'v'$ are compatible if and only if the free space of uv is contained in one of three rectilinear subpolygons of P defined by (a'_1, a'_2, a'_3), and the free space of $u'v'$ is completely contained in one of three rectilinear subpolygons of P defined by (a_1, a_2, a_3).*

Proof. Let P_i and P'_i, for $i \in \{1, 2, 3\}$, be the rectilinear subpolygons of P defined by (a_1, a_2, a_3) and (a'_1, a'_2, a'_3), respectively. We can assume that all a'_i are contained in ∂P_i for some $i \in \{1, 2, 3\}$; otherwise, a roof edge associated with uv and a roof edge associated with $u'v'$ cross, for which there is no realistic roof containing uv and $u'v'$. This also implies that all a_i are contained in $\partial P'_i$ for some $i \in \{1, 2, 3\}$. Consider the case that all a_i are contained in $\partial P'_1$, and therefore all a'_i are contained in ∂P_1. Assume to the contrary that the free space of uv is not contained in any of P'_1, P'_2 and P'_3, as depicted in Figure 11(a). This implies that r_1 intersects $\partial P'_1$ and $y(a_1) - y(a'_1) < x(a_3) - x(a_1)$. Let s and s' denote the two peak points of uv and $u'v'$, respectively. Let p be the point $h \cap (a'_1s' \cup s'u' \cup u'v')$, where h is the plane through s and parallel to the yz-plane. Since $y(s) < (y(a_1) + y(a'_1))/2$, we have $y(s) - y(p) < z(s) - z(p)$ and therefore the portion of $R \cap h$ from s to p must have an edge of slope larger than 1, which is not allowed in a realistic roof. The remaining two cases that all a_i are contained in either $\partial P'_2$ or $\partial P'_3$ can also be shown to make uv and $u'v'$ not compatible by using a similar argument.

Suppose now that the free space of uv is contained in one of three rectilinear subpolygons of P defined by (a'_1, a'_2, a'_3), and the free space of $u'v'$ is completely contained in one of three rectilinear subpolygons of P defined by (a_1, a_2, a_3). We show how to construct a realistic roof with uv and $u'v'$. Without loss of generality, we assume that P_1 contains a'_1, a'_2 and a'_3. Let P_{11}, P_{12} and P_{13} denote the rectilinear subpolygons of P_1 defined by (a'_1, a'_2, a'_3). Now we have five rectilinear subpolygons $P_{11}, P_{12}, P_{13}, P_2$ and P_3 of P. By taking the upper envelope of the roofs $R^*(P_{11}), R^*(P_{12}), R^*(P_{13}), R^*(P_2)$ and $R^*(P_3)$, we can get a realistic roof which contains uv and $u'v'$. □

Lemma 8 *Let uv be a half-open valley and let (a'_1, a'_2) be a candidate pair of footholds for an open valley $u'v'$. Two valleys uv and $u'v'$ are compatible if and only if the smallest axis-aligned rectangle containing a'_1 and a'_2 does not cross the free space of uv properly.*

Proof. Let P_i, for $i \in \{1, 2, 3\}$, be the rectilinear subpolygons of P defined by the triple (a_1, a_2, a_3) of footholds of uv. We can assume that a'_1 and a'_2 are contained in ∂P_i for some $i \in \{1, 2, 3\}$; otherwise, a roof edge associated with uv and a roof edge associated with $u'v'$ cross, for which there is no realistic roof containing uv and $u'v'$. Let B denote the smallest axis-aligned rectangle containing a'_1 and a'_2. If a'_1 and a'_2 are contained in ∂P_2 or ∂P_3, then B does not cross the free space of uv properly.

Suppose that a'_1 and a'_2 are contained in ∂P_1 and B crosses the free space of uv properly, as depicted in Figure 11(b). Let p be the point $h \cap (a'_1u' \cup u'v' \cup v'a'_2)$, where h is the plane through s and parallel to the yz-plane. Since $y(s) < (y(a_1) + y(a'_1))/2$, we have $y(s) - y(p) < z(s) - z(p)$ and therefore the portion of $R \cap h$ from s to p must have an edge of slope larger than 1, which is not allowed in a realistic roof.

Suppose now that B does not cross the free space of uv properly. We show how to construct a realistic roof with uv and $u'v'$. Ahn et al. [1] showed how to construct a realistic roof R over P' with a candidate

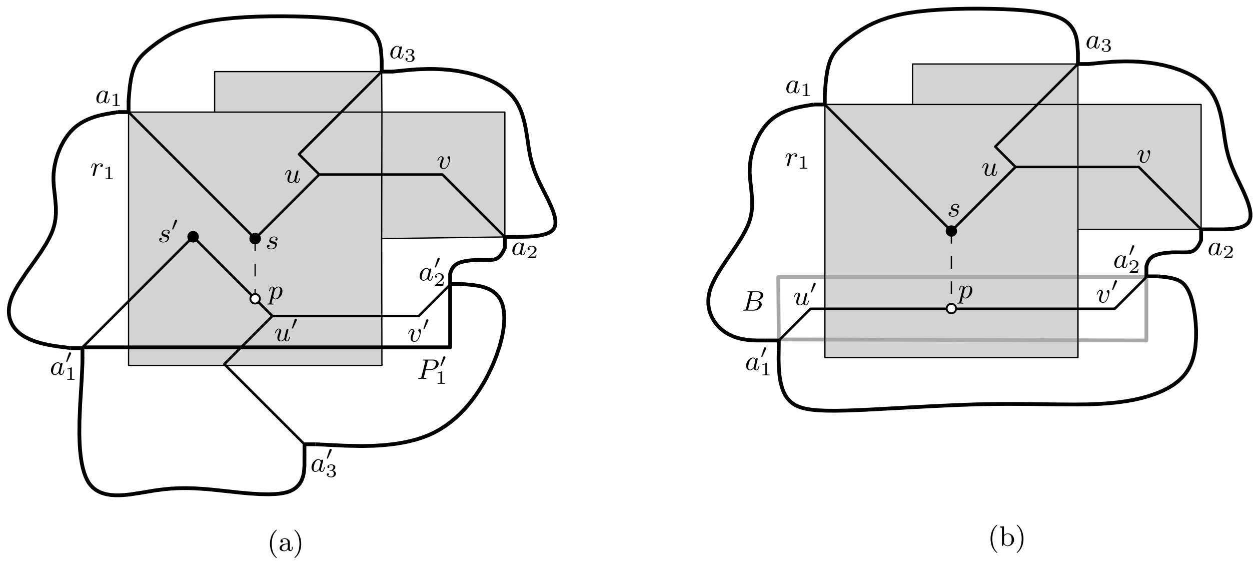

Figure 11: (a) The free space of uv (gray) crosses $\partial P_1'$. Then we have $y(s) - y(p) < z(s) - z(p)$, for which we cannot construct a realistic roof. (b) The free space of uv crosses B properly. Then we have $y(s) - y(p) < z(s) - z(p)$, for which we cannot construct a realistic roof.

pair of footholds (a_1', a_2') for an open valley $u'v'$: Divide P' into two subpolygons by a chain $a_1'\overline{u'} \cup \overline{u'v'} \cup \overline{v'}a_2'$ and let P_1' be the union of one subpolyon and B and P_2' be the union of the other subpolygon and B; Then take the upper envelope of $R^*(P_1') \cup R^*(P_2')$.

Without loss of generality, we assume that P_1 contains a_1' and a_2'. Chain $a_1'\overline{u'} \cup \overline{u'v'} \cup \overline{v'}a_2'$ divides P_1 into two subpolygons. Let P_{11} be the union of one subpolygon and B, and let P_{12} be the union of the other subpolygon and B. Then both P_{11} and P_{12} are rectilinear polygons. Now we have four rectilinear subpolygons P_{11}, P_{12}, P_2 and P_3 of P. By taking the upper envelope of the roofs $R^*(P_{11}), R^*(P_{12}), R^*(P_2)$ and $R^*(P_3)$, we can get a realistic roof which contains uv and $u'v'$. □

Let V be a set of candidate pairs of footholds and candidate triples of footholds. If every pair of elements of V satisfies Lemma 6 or Lemma 7 or Lemma 8, we can find a unique realistic roof R whose valleys correspond to V. Also, we call such V a *compatible valley set* of P. We conclude this section with the following theorem.

Theorem 1 *Let P be a rectilinear polygon with n vertices and V be a compatible valley set of k candidate pairs of footholds and l candidate triples of footholds with respect to P. Then there exists a unique realistic roof R whose valleys correspond to V. In addition, there exist $k+2l+1$ rectilinear subpolygons $P_1, \ldots, P_{k+2l+1}$ of P such that*

1. $\bigcup_{i=1}^{k+2l+1} P_i = P$.
2. *R coincides with the upper envelope of $R^*(P_i)$'s, for all $i = 1, \ldots, k+2l+1$.*

5 The Number of Realistic Roofs

We give an upper bound of the number of possible realistic roofs over P in terms of n. For this, we need a few technical lemmas.

Lemma 9 *Let (a_1, a_2, a_3) be a candidate triple of footholds for a half-open valley, where a_1 and a_2 have opposite orientations. Then (a_1, a_2) is also a candidate pair of footholds.*

Proof. The candidate triple (a_1, a_2, a_3) admits a half-open valley uv. The free space of uv contains the smallest axis-aligned rectangle containing a_1 and a_2, so a_1 and a_2 admit an open valley. ◻

Lemma 10 *Let (a_1, a_2, a_3) be a candidate triple of footholds for a half-open valley uv, where a_1 and a_2 have opposite orientations. If a candidate pair (a_4, a_5) of footholds for an open valley is compatible with (a_1, a_2, a_3), then there is no half-open valley with footholds (a_3, a_4, a_5).*

Proof. Without loss of generality, assume that the three reflex vertices a_1, a_2, a_3 and the valley uv are oriented and placed as in Figure 12(a). By Lemma 8, both a_4 and a_5 must be contained in one of three rectilinear subpolygons P_1, P_2 and P_3 of P defined by uv. Assume to the contrary that (a_3, a_4, a_5) is a candidate triple of footholds for a half-open valley $u'v'$. There are four cases for (a_4, a_5) as follows.

If $a_4, a_5 \in \partial P_3$, there is only one possible configuration as depicted in Figure 12(b). By some careful case analysis, we have $d_x(3,5) > d_x(3,4) > d_y(3,4)$, which makes a_4 be contained in the interior of the free space of $u'v'$. In case that $a_4, a_5 \in \partial P_2$, there is no possible configuration. Finally, consider the case that $a_4, a_5 \in \partial P_1$. There are two possible configurations. When $x(a_5) < x(a_4) < x(a_1)$ as depicted in Figure 12(c), we have $d_x(3,5) > d_x(3,1) > d_y(3,1)$, which makes a_1 be contained in the interior of the free space of $u'v'$. When $x(a_5) < x(a_1) < x(a_4)$ as depicted in Figure 12(d), we have $d_y(3,4) > d_x(3,1) > d_y(3,1)$, which again makes a_1 be contained in the interior of the free space of $u'v'$. ◻

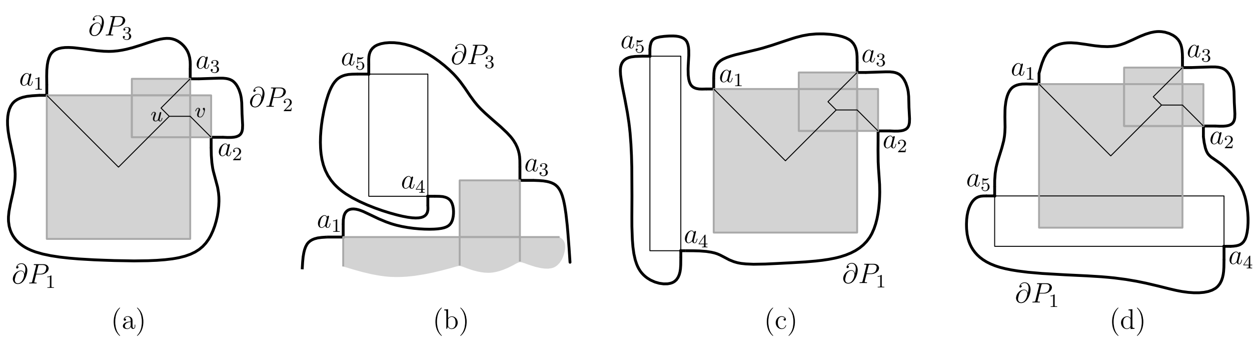

Figure 12: Illustration of the proof of Lemma 10. Gray regions are free spaces.

Based on the two previous lemmas, we give an upper bound on the number of realistic roofs over P.

Theorem 2 *Let P be a rectilinear polygon with n vertices. There are at most $1.3211^m \binom{m}{\lfloor \frac{m}{2} \rfloor}$ distinct realistic roofs over P, where $m = \frac{n-4}{2}$.*

Proof. Let R be a realistic roof over P with a half-open valley uv. By Lemma 9, we can get an open valley $u'v'$ induced by two footholds of uv that have opposite orientations. Therefore, we can get a new realistic roof by replacing uv with $u'v'$. By repeating this process, we can get a realistic roof R' which does not contain any half-open valleys. It means that for any realistic roof R over P, there exists a unique realistic roof R' which has no half-open valleys. We can get the number of distinct realistic roofs over P with two steps: counting the number of realistic roofs R' over P which has no half-open valleys and counting the number of realistic roofs R which can be transformed to each R' by replacing its half-open valleys with open valleys.

Ahn et al. [1] gave an upper bound on the number of realistic roofs R' over P which have no half-open valleys, which is $\binom{m}{\lfloor \frac{m}{2} \rfloor}$, where $m = \frac{n-4}{2}$. We calculate the number of realistic roofs R over P corresponding to each R'. Suppose that R' contains k open valleys, $u_1v_1, u_2v_2, \ldots, u_kv_k$. P has $m - 2k$ reflex vertices that are not used as footholds of these open valleys. Let us call these reflex vertices *free vertices* of R'. By Lemma 10, each free vertex can make a half-open valley with at most one open valley. Let x_i, $1 \le i \le k$, be the number of free vertices of R' that can make a half-open valley with u_iv_i. Then the number of realistic roofs that can be transformed to R' is at most $(x_1+1)(x_2+1)\cdots(x_k+1)$, where $x_1+x_2+\ldots+x_k \leq m-2k$. From the inequality of arithmetic and geometric means, we can get

$$\begin{aligned}(x_1+1)(x_2+1)\cdots(x_k+1) &\leq (\frac{x_1+x_2+\ldots+x_k+k}{k})^k \\ &\leq (\frac{m-k}{k})^k \\ &= ((\frac{m}{k}-1)^{\frac{k}{m}})^m.\end{aligned}$$

For a positive real number x, we have $\sup\{(x-1)^{\frac{1}{x}}\} \approx 1.3210998$, so $((\frac{m}{k}-1)^{\frac{k}{m}})^m < 1.3211^m$. Therefore, we can get at most 1.3211^m different realistic roofs over P corresponding to each R', and the total number of distinct realistic roofs over P is at most $1.3211^m\binom{m}{\lfloor \frac{m}{2} \rfloor}$. □

In the case of an orthogonally convex rectilinear polygon P, we can get a better upper bound on the number of realistic roofs over P. An orthogonally convex rectilinear polygon is a simple rectilinear polygon such that for any line segment parallel to any of the coordinate axes connecting two points lying within the polygon lies completely within the polygon. The boundary of an orthogonally convex rectilinear polygon consists of four *staircases* [12]. See Figure 13.

From Lemma 4, a half-open valley uv has three footholds a_i, a_j and a_k, which are reflex vertices of P in mutually different orientations, and therefore each of which is contained in a different staircase. Also from Lemma 7, a realistic roof of P containing uv can contain only one additional half-open valley $u'v'$ because only one chain of $\partial P \setminus \{a_i, a_j, a_k\}$ can have three reflex vertices of mutually different orientations. Therefore, all realistic roofs over an orthogonally convex rectilinear polygon can have at most two half-open valleys as shown in Figure 13.

We give an upper bound on the number of realistic roofs over P as we did in the proof of Theorem 2. Let R' be a realistic roof over P which has no half-open valleys and let k denote the number of open valleys $u_1v_1, u_2v_2, \ldots, u_kv_k$ in R'. Let x_i denote the number of free vertices of P which can induce a half-open valley with u_iv_i. The number of realistic roofs that can be transformed to R' is at most $\sum_{i,j} x_ix_j \leq \binom{k}{2}m^2 \leq m^4$. Therefore, the number of distinct realistic roofs over P is at most $m^4\binom{m}{\lfloor \frac{m}{2} \rfloor}$.

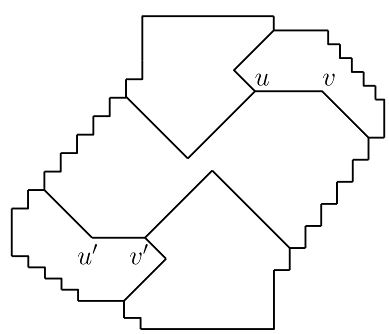

Figure 13: An orthogonally convex rectilinear polygon P with two half-open valleys uv and $u'v'$

Theorem 3 *Let P be an orthogonally convex rectilinear polygon with n vertices. There are at most $m^4\binom{m}{\lfloor \frac{m}{2} \rfloor}$ distinct realistic roofs over P, where $m = \frac{n-4}{2}$.*

6 Algorithm

In this section, we will present an algorithm that generates all possible realistic roofs over a given rectilinear polygon P. Ahn et al. [1] suggested an efficient algorithm that generates all realistic roofs which do not contain half-open valleys. Let GenerateOpenValleys denote the algorithm. GenerateOpenValleys spends $O(n^4)$ time in preprocessing and generates realistic roofs one by one in $O(1)$ time each. Our algorithm also spends $O(n^4)$ time in preprocessing: P has $O(n^3)$ triples and $O(n^2)$ pairs of reflex vertices, and checking whether each triple and pair is a candidate triple or candidate pair takes $O(n)$ time. And then, we create an empty list L_{uv} of reflex vertices for each candidate pair of uv and add a reflex vertex a_i to L_{uv} if a_i and the footholds of uv form a candidate triple.

Our algorithm works as follows. It runs GenerateOpenValleys and gets a realistic roof R with k open valleys $u_1v_1, \dots, u_kv_k$. A pair (a_i, a'_i) of footholds corresponding to $u_iv_i,\ 1 \le i \le k$, has a list $L_{u_iv_i}$ of reflex vertices. Our algorithm either chooses a reflex vertex w_i from $L_{u_iv_i}$ or not. Let V_O denote the set of pairs of footholds for which no reflex vertex is chosen, and let V_H denote the set of triples (a_i, a'_i, w_i) such that a reflex vertex w_i is chosen for (a_i, a'_i). If no reflex vertex is chosen for any pair (a_i, a'_i) of footholds, that is, $V_H = \emptyset$, then the realistic roof with open valleys of V is exactly R. Otherwise, our algorithm checks whether every pair of valleys in $V_O \cup V_H$ is compatible as follows. Suppose that we have already checked the compatibility of pairs of valleys in $V_O \cup V_H$ and let N_i denote the number of valleys in $(V_O \cup V_H) \setminus \{(a_i, a'_i, w_i)\}$ incompatible with (a_i, a'_i, w_i).

When we replace w_i with another reflex vertex w'_i in $L_{u_iv_i}$, we compute the compatibility between (a_i, a'_i, w'_i) and each valley in $(V_O \cup V_H) \setminus \{(a_i, a'_i, w_i)\}$ only and update N_i. This can be done in $O(k)$ time. If $\sum_{i=1}^{k} N_i = 0$, every pair of valleys in $V_O \cup V_H$ is compatible, and therefore there is a roof with valleys of $V_O \cup V_H$. Therefore, our algorithm finds all realistic roofs correspond to P in $O(m1.3211^m)$ time.

Theorem 4 *Given a rectilinear polygon P with n vertices, m of which are reflex vertices, after $O(n^4)$-time preprocessing, all the compatible sets of P can be enumerated in $O(m1.3211^m \binom{m}{\lfloor \frac{m}{2} \rfloor})$ time.*

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