Realistic Roofs without Local Minimum Edges over a Rectilinear Polygon

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Abstract
Computing all possible roofs over a given ground plan is a common task in automatically reconstructing a three dimensional building. In 1995, Aichholzer et al. proposed a definition of a roof over a simple polygon \(P\) in the \(xy\)-plane as a terrain over \(P\) whose faces are supported by planes containing edges of \(P\) and making a dihedral angle \(\frac{\pi}{4}\) with the \(xy\)-plane. This definition, however, allows roofs with faces isolated from the boundary of \(P\) and local minimum edges inducing pools of rainwater. Very recently, Ahn et al. introduced “realistic roofs” over a rectilinear polygon with \(n\) vertices by imposing two additional constraints under which no isolated faces and no local minimum vertices are allowed. Their definition is, however, restricted and excludes a number of roofs with no local minimum edges. In this paper, we propose a new definition of realistic roofs over a rectilinear polygon that corresponds to the class of roofs without isolated faces and local minimum edges. We investigate the geometric and combinatorial properties of realistic roofs and show that the maximum possible number of distinct realistic roofs over a rectilinear \(n\)-gon is at most \(1.3211^m \left(\frac{m^2}{2}\right)\), where \(m = \frac{n-4}{2}\). We also present an algorithm that generates all combinatorial representations of realistic roofs.

1 Introduction
A common task in automatically reconstructing a three dimensional city model from its two dimensional map is to compute all the possible roofs over the ground plans of its buildings [4, 5, 11, 9, 10, 13]. For instance, Figure 1(a) shows a ground plan of a building in a perspective view, which is the union of two overlapping rectangles. The roof in Figure 1(b) can be constructed by building a roof over each rectangle and taking the upper envelope of the two roofs. The roof in Figure 1(c) can be constructed by shrinking the ground plan at a constant speed while moving it along vertically upward at a constant speed. Note that the vertical projection of the roof coincides with the the straight skeleton of the ground plan [2, 3].

For some applications, a correct or reasonable roof over a building is chosen from its set of possible roofs by considering some additional information such as its satellite images.

Aichholzer et al. [2] defined a roof over a simple (not necessarily rectilinear) polygon in the \(xy\)-plane as a terrain over the polygon such that the polygon boundary is contained in the terrain and each face of the terrain is supported by a plane containing at least one polygon edge and making a dihedral angle \(\frac{\pi}{2}\) with the \(xy\)-plane. This definition, however, is not tight enough that it allows roofs with faces isolated from the boundary of the polygon (Figure 2(a)) and local minimum edges (Figure 2(b)) which are undesirable for some practical reasons – for example, a local minimum edge serves as a pool of rainwater, which can cause damage to the roof. Note that a pool of rainwater on a roof always contains a local minimum edge or vertex.

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Figure 1: A rectilinear ground plan and two different roofs over the plan in a perspective view.

Figure 2: (a) A roof with isolated faces $f$ and $f'$. (b) A roof with a local minimum edge $e$. (c) Not a realistic roof according to Definition 1; vertex $u$ has no adjacent vertex that is lower than itself.

1.1 Related work

Brenner [5] designed an algorithm that computes all the possible roofs over a rectilinear polygon, but no polynomial bound on its running time is known. Recently, Ahn et al. [1] introduced “realistic roofs” over a rectilinear polygon $P$ with $n$ vertices by imposing two additional constraints to the definition of “roofs” by Aichholzer et al. [2] as follows.

Definition 1 ([1]) A realistic roof over a rectilinear polygon $P$ is a roof over $P$ satisfying the following constraints. 

C1. Each face of the roof is incident to at least one edge of $P$.

C2. Each vertex of the roof is higher than at least one of its neighboring vertices.

They showed some geometric and combinatorial properties of realistic roofs, including a connection to the straight skeleton [2, 3, 7, 6, 8]. Consider a roof $R^*(P)$ over $P$ constructed by a shrinking process, where all of the edges of $P$ move inside, being parallel to themselves, with the same speed while moving upward along the $z$-axis with the same speed. Aichholzer et al. [2] showed that $R^*(P)$ is unique and its projection on the $xy$-plane is the straight skeleton of $P$. Ahn et al. [1] showed that $R^*(P)$ is the pointwise highest realistic roof over $P$. From the fact that $R^*(P)$ does not have a “valley”, Ahn et al. [1] suggested a way of constructing another realistic roof over $P$ different to $R^*(P)$ by adding a set of “compatible valleys” to $R^*(P)$. They showed that the number of realistic roofs lies between 1 and $(m ⌊ m^2 ⌋)$ where $m = n - 4$, and presented an output sensitive algorithm generating all combinatorial representations of realistic roofs over $P$ in $O(1)$ amortized time per roof, after $O(n^4)$ preprocessing time.

1.2 Our results

Constraint $C1$ in Definition 1 was introduced to exclude roofs with isolated faces and constraint $C2$ was introduced to avoid pools of rainwater. However, $C2$ is restricted and excludes a large number of roofs containing no local minimum edges. For example, the roof in Figure 2(c) is not realistic according to
Definition 1 though rainwater can be drained well along $uv$. Therefore, Definition 1 by Ahn et al. [1] does describe only a subset of “realistic” roofs.

We introduce a new definition of realistic roofs by replacing $C_2$ of Definition 1 with a relaxed one that excludes roofs with local minimum edges only.

**Definition 2** A realistic roof over a rectilinear polygon $P$ is a roof over $P$ satisfying the following constraints.

$C_1$. Each face of the roof is incident to at least one edge of $P$.

$C_2'$. For each roof edge $uv$, $u$ or $v$ is higher than at least one of its neighboring vertices.

From now on, we use Definition 2 for realistic roofs unless stated explicitly. Our definition corresponds to the class of roofs without isolated faces, local minimum edges and local minimum vertices exactly.

Our main results are threefold:

1. We provide a new definition of “realistic roofs” that corresponds to the real-world roofs and investigate geometric properties of them.
2. We show that the maximum possible number of realistic roofs over a rectilinear $n$-gon is at most $1.3211^m\left(\frac{m}{\pi}\right)$, where $m = \frac{n-4}{2}$.
3. We present an algorithm that generates all combinatorial representations of realistic roofs over a rectilinear $n$-gon. Precisely, it generates a roof whose vertices are all open, that is, every vertex is higher than at least one of its neighboring vertices in $O(1)$ time after $O(n^4)$ preprocessing time [1]. For each such roof $R$, it generates $O(1.3211^m)$ realistic roofs in time $O(m1.3211^m)$ by adding edges on $R$.

## 2 Preliminary

For a point $p$ in $\mathbb{R}^3$, we denote by $x(p), y(p),$ and $z(p)$ the $x$-, $y$-, and $z$-coordinate of $p$, respectively. We denote by $\overline{p}$ the orthogonal projection of $p$ onto the $xy$-plane. A line through $\overline{p}$ parallel to the $x$-axis, and another line through $\overline{p}$ parallel to the $y$-axis together divide the $xy$-plane into four regions, called quadrants of $\overline{p}$, each bounded by two half-lines. For a point $q$ in a roof $R$, let $D(q)$ be the axis-parallel square centered at $\overline{q}$ with side length $2z(q)$.

We denote by $\partial P$ the boundary of $P$ and by edge($f$) the edge of $\partial P$ incident to a face $f$ of a roof.

**Lemma 1 ([1])** Let $R$ be a roof over a rectilinear polygon $P$. The followings hold.

(a) For any point $p \in R$, $z(p)$ is at most the $L_\infty$ distance from $\overline{p}$ to its closest point in $\partial P$. Therefore, we have $D(p) \subseteq P$.

(b) For each edge $e$ of $P$, there exists a unique face $f$ of $R$ incident to $e$.

(c) Every face $f$ of $R$ is monotone with respect to the line containing edge($f$).

Consider the boundary $\partial f$ of $f$. According to property (c) of Lemma 1, $\partial f$ consists of exactly two chains monotone with respect to the line containing edge($f$).

An edge $e$ of a realistic roof $R$ over $P$ is convex if the two faces incident to $e$ make a dihedral angle below $\pi$, and reflex otherwise. A convex edge is called ridge if it is parallel to the $xy$-plane. A reflex edge is called a valley if it is parallel to the $xy$-plane.

## 3 Valleys of a Realistic Roof

In this section, we investigate local structures of realistic roofs. Ahn et al. [1] showed five different configurations of end vertices that a ridge can have under Definition 1. They also showed that vertices which are not incident to a valley or a ridge are degenerated forms of valleys or ridges. Since replacing constraint $C_2$ with $C_2'$ does not affect ridges, we care about only valleys.

We define three types of valleys and describe their structures that a realistic roof can have. We call a vertex of a roof open if it is higher than at least one of its neighboring vertices connected by roof edges, and
closed otherwise. We call a valley open if both end vertices are open, half-open if one end vertex is open and the other is closed, and closed if both end vertices are closed. For instance, the valley uv in Figure 2(c) has an open end vertex v and a closed end vertex u, and therefore it is half-open.

By Definition 3, a realistic roof can contain open and half-open valleys but it does not contain closed valleys. Ahn et al. [1] showed that each open valley always has the same structure as st in Figure 2(c). More specifically, they first showed that there are only 5 possible configurations near an end vertex of a valley which satisfy the roof constraints such as the monotonicity of a roof, and the slope and orientations of faces as illustrated in Figure 3. Then they showed that an open valley must have both end vertices of configuration (v1) only and oriented symmetrically along the valley. Otherwise, an end vertex of the valley becomes a local minimum or a face f incident to the valley is not monotone with respect to the line containing edge(f) contradicting Lemma 1(c). They also observe that each end vertex of an open valley is connected to a reflex vertex of P by a reflex edge. We call such a reflex vertex a foothold of the open valley. Note that two footholds a and a’ of an open valley uv are opposite corners of B_{aa’} and B_{aa’} \{a,a’\} is contained in the interior of P, and uv coincides with the ridge of R^*(B_{aa’}), where B_{aa’} denote the smallest axis-aligned rectangle containing a and a’.

Figure 3: Five possible configurations around a vertex u of a valley uv shown by Ahn et al. [1], where rf denotes a reflex edge and cv denotes a convex edge. Each convex or reflex edge is oriented from the endpoint with smaller z-coordinate to the other one with larger z-coordinate.

In the following we investigate the structure of a half-open valley that a realistic roof can have. It is not difficult to see that the open end vertex is always of configuration (v1); and any end vertex of the other configurations cannot have a lower neighboring vertex. We will show that every closed end vertex of a valley is always of configuration (v2). For this, we need a few technical lemmas.

Lemma 2 Let uv be a valley and uv’ be a convex edge incident to u. Also, let ℓ be the line in the xy-plane passing through v and orthogonal to uv. Then the face f incident to both uv and uv’ has edge(f) in the half-plane of ℓ in the xy-plane not containing v.

Proof. Figure 3 shows all possible configurations that an end vertex u of a valley uv has. Since uv’ is convex, v’ is strictly higher than u and uv’ makes an angle 45° with uv in all cases. Then the lemma follows from the monotonicity property (c) of Lemma 1.

Imagine that a face f is incident to a valley uv and two convex edges one of which is incident to u and the other to v. This is only possible when both convex edges lie in the same side of the plane containing uv and parallel to the z-axis, because of the monotonicity of a roof, and the slope and orientations of faces. Since both convex edges make an angle 45° with uv in their projection on the xy-plane, f cannot have a ground edge by Lemma 2 that is, f is isolated.

Lemma 3 Let uv be a half-open valley of a realistic roof where u is closed and v is open. Then v is of configuration (v1) and u is of configuration (v2).

Proof. If u is of configuration (v3), then one of two faces incident to uv becomes isolated by Lemma 2. If u is of configuration (v5), there always is another valley uv’ that is orthogonal to uv and has a closed corner at u of configuration (v3) as shown in Figure 3. Therefore one of faces incident to uv’ is isolated.
Assume now that \( u \) is of configuration (v4). Then there always is another valley \( uv' \) orthogonal to \( uv \). Therefore, we need to check two connected valleys \( uv \) and \( uv' \) simultaneously. Figure 4 illustrates all possible combinations of these two valleys. For cases (a) and (b), there is an isolated face incident to \( uv \) or \( uv' \). For case (c), let \( f \) and \( f' \) be the faces incident to \( uv \) and \( uv' \), respectively, sharing the reflex edge incident to \( u \) as shown in Figure 4(c). By Lemma 2, \( \text{edge}(f) \) must lie in the top right quadrant of \( \pi \) and \( \text{edge}(f') \) must lie in the bottom left quadrant of \( \pi \) in the \( xy \)-plane. This is, however, not possible unless \( f \) or \( f' \) violates the monotonicity property (c) of Lemma 1.

Figure 4: Three possible combinations around a (v4) type vertex.

The only remaining closed end vertex is of configuration (v2). Figure 5 shows a half-open valley \( uv \) with \( u \) of configuration (v1) and \( v \) of configuration (v2).

Figure 5: A half-open valley \( uv \) must be connected to three reflex vertices \( a_1, a_2 \) and \( a_3 \) of \( P \) via five reflex edges. We call the vertex \( s \) which is incident to \( rf_1 \) and \( rf_4 \) the peak point of \( uv \).

Now we are ready to describe the structure of a half-open valley. In the following, we show that a half-open valley always has the same structure on a realistic roof as in Figure 5. Specifically, a half-open valley \( uv \) is associated with five reflex edges of the roof and three reflex vertices of \( P \) which have mutually different orientations. We call the three reflex vertices of \( P \) that induce a half-open valley the footholds of the valley.

Open vertex \( v \) to foothold \( a_2 \) Suppose that \( rf_3 \) in Figure 5 is not connected to a reflex vertex of \( P \). Then \( rf_3 \) must be incident to another half-open valley \( u'v' \), because a closed vertex of configuration (v2) is the only roof vertex that can have such a reflex edge. By Lemma 3, there are four possible cases and they are illustrated in Figure 6.

In case (a), face \( f_1 \) is isolated by the monotonicity property (c) of Lemma 1. In case (b), by the monotonicity of \( f_1 \), \( \text{edge}(f_1) \) must lie in the top left quadrant of \( \pi \) in the \( xy \)-plane. This implies that \( \text{edge}(f_2) \) must lie in the top right quadrant of \( \pi \), and \( \text{edge}(f_3) \) must lie in the bottom right quadrant of \( \pi \) in
Figure 6: Four possible cases of two half-open valleys, $uv$ and $u'v'$, connected by reflex edge $u'v$.

the $xy$-plane. However, by the monotonicity of $f_4$, edge($f_4$) must lie in the top left quadrant of $v'$, and this is not possible unless $f_3$ or $f_4$ violates the monotonicity property (c) of Lemma 1. In case (c), edge($f_1$) must lie in the top left quadrant and edge($f_3$) must lie in the bottom left quadrant of $u$ in the $xy$-plane. Then $f_1$ or $f_3$ violates the monotonicity property. In case (d), edge($f_1$) must lie in the bottom right quadrant and edge($f_2$) must lie in the top left quadrant of $u'$ in the $xy$-plane. This is, however, not possible unless $f_1$ or $f_2$ violates the monotonicity property. Therefore, $v$ must be connected to a reflex vertex $a_2$ of $P$ via $rf_3$.

Figure 7: When $rf_1$ is connected to either (a) an open valley $u'v'$ or (b) a half-open valley $u'v'$.

Closed vertex $u$ to footholds $a_1$ and $a_3$. We show that $u$ is connected to foothold $a_1$ via two reflex edges $rf_1$ and $rf_4$. Note that the end vertex of $rf_1$ other than $u$ is an end vertex (of configuration (v1)) of a valley or a ridge.

When $rf_1$ is connected to an open valley $u'v'$, both $uv$ and $u'v'$ are incident to a face $f_1$, which is isolated. See Figure 7(a). If $u'v'$ is a half-open valley, then one of two faces incident to $u'v'$ violates the monotonicity (c) of Lemma 1. See Figure 7(b).

When $rf_1$ is connected to a ridge, there is another reflex edge $rf_4$ incident to the ridge. Suppose that $rf_4$ is not connected to a reflex vertex of $P$. Then $rf_4$ must be incident to another half-open valley $u'v'$. Figure 8 shows all four possible cases, but none of them can be constructed in a realistic roof: either a face is isolated (cases (a) and (c)) or at least one face violates the monotonicity (c) of Lemma 1 (cases (b) and (d)). Therefore, $u$ must be connected to a reflex vertex $a_1$ of $P$ via two reflex edges $rf_1$ and $rf_4$.

In a similar way, we can show how $u$ is connected to foothold $a_3$ via two reflex edges $rf_2$ and $rf_5$.

**Lemma 4** Let $uv$ be a half-open valley where $u$ is closed and $v$ is open. Then $uv$ is associated with three reflex vertices of $P$ that have mutually different orientations as shown in Figure 5.

### 4 Realistic Roofs with Half-Open Valleys

From Lemma 4, we know that a half-open valley is associated with three reflex vertices that have mutually different orientations. In the following we investigate a condition under which three reflex vertices $a_1, a_2,$ and $a_3$ with mutually different orientations can induce a half-open valley.
Let \( d_x(i, j) := x(a_i) - x(a_j) \) and \( d_y(i, j) := y(a_i) - y(a_j) \). Without loss of generality, we assume that these three vertices are oriented and placed as in Figure 5. That is, we have \( d_x(3, 1), d_x(2, 3), d_y(1, 2), d_y(3, 1) > 0 \).

We define two squares and one rectangle in the \( xy \)-plane to determine whether these three reflex vertices form a half-open valley. Let \( r_1 \) be the square with \( a_1 \) on its top left corner and side length \( d_x(3, 1) \). Let \( r_2 \) be the rectangle with \( a_2 \) on its bottom right corner with height \( d_y(1, 2) \) and width \( d_y(1, 2) + d_x(2, 3) \). Finally, let \( r_3 \) be the square with \( a_3 \) on its top right corner and side length \( d_y(3, 2) \). Note that these three rectangles overlap each other and have a nonempty common intersection.

We define three rectilinear subpolygons of \( P \) along \( r_1, r_2, \) and \( r_3 \) as follows. Let \( P' := P \setminus (r_1 \cup r_2 \cup r_3) \).

Let \( P_1 \) denote the union of \( r_1 \cup r_2 \) and the components of \( P' \) incident to the portion of \( \partial P \) from \( a_1 \) to \( a_2 \) in a counterclockwise direction (Figure 9(a)). Let \( P_2 \) denote the union of \( r_2 \cup r_3 \) and the components of \( P' \) incident to the portion of \( \partial P \) from \( a_2 \) to \( a_3 \) in a counterclockwise direction (Figure 9(b)). Let \( P_3 \) denote the union of \( r_1 \cup r_3 \) and the components of \( P' \) incident to the portion of \( \partial P \) from \( a_3 \) to \( a_1 \) in a counterclockwise direction (Figure 9(c)).

**Figure 8:** When \( rf_4 \) is connected to another half-open valley \( u'v' \).

**Lemma 5** There is a realistic roof with a half-open valley induced by reflex vertices \( a_1, a_2, \) and \( a_3 \) of \( P \) if and only if \((r_i \setminus a_i) \cap \partial P = \emptyset\), for all \( i \in \{1, 2, 3\} \).
Proof. Let $uv$ be the half-open valley of a realistic roof $R$ induced by $a_1, a_2$ and $a_3$. We know that $uv$ is connected to $a_1, a_2$ and $a_3$ via five reflex edges as shown in Figure 9. Note that $r_1 = D(s), r_3 = D(t)$, and $r_2 = \bigcup_{p \in uv} D(p)$. Therefore, $r_i \subseteq P$ for all $i \in \{1, 2, 3\}$. Let $S_\varepsilon$ denote the set of points on $R$ in the $\varepsilon$-neighborhood of $s$ for small $\varepsilon > 0$. By property (a) of Lemma 1, we have $D(p) \subseteq P$ for every $p \in S_\varepsilon$. Since $s$ is an end vertex of a ridge and it is incident to two reflex edges, $\bigcup_{p \in S_\varepsilon} D(p)$ contains $\partial r_1$ in its interior, except $a_1$ and the top right corner of $r_1$. The top right corner of $r_1$ coincides with the top right corner of $D(u)$, and there is a point $q$ on $R$ near $u$ such that $D(q)$ contains the top right corner of $r_1$ in its interior. By using a similar argument, we can show that $(r_3 \setminus a_3) \cap \partial P = \emptyset$. For $r_2$, let $U_\varepsilon$ denote the set of points on $R$ in the $\varepsilon$-neighborhood of $uv$ for small $\varepsilon > 0$. Since $uv$ is a half-open valley, $\bigcup_{p \in U_\varepsilon} D(p)$ contains $\partial r_2$ in its interior, except $a_2$.

Now assume that $(r_i \setminus a_i) \cap \partial P = \emptyset$ for all $i \in \{1, 2, 3\}$. We will show that the upper envelope of $R^*(P_1) \cup R^*(P_2) \cup R^*(P_3)$ forms a realistic roof $R$ over $P$ which contains the unique half-open valley $uv$ induced by $a_1, a_2$ and $a_3$. Since $P_1$ and $P_2$ both contain $r_2$, $R^*(P_1)$ and $R^*(P_2)$ intersect along $a_2v$ and $uv$. Likewise, $P_2$ and $P_3$ both contain $r_3$, so $R^*(P_2)$ and $R^*(P_3)$ intersect along $a_3t$ and $ut$. Finally, $P_1$ and $P_3$ both contain $r_1$, so $R^*(P_1)$ and $R^*(P_3)$ intersect along $a_1s$ and $us$. Therefore $uv$ and its five associated reflex edges appears on $R$.

It remains to show that every face $f$ on the upper envelope of $R^*(P_1) \cup R^*(P_2) \cup R^*(P_3)$ is not isolated and monotone along the line containing edge($f$). Since all faces in $R^*(P_i)$, for all $i \in \{1, 2, 3\}$ satisfy the condition, it suffices to consider only faces incident to $uv$ and its five associated reflex edges.

Consider the face $f_1$ that is incident to $uv, rf_1$ and $rf_4$. Since $r_1$ touches $\partial P$ only at $a_1$, there exists a rectangle $r'_1 \subseteq P_1$ that contains $r_1$ and whose boundary contains the top side of $r_1$ only. Since $r_2$ touches $\partial P$ only at $a_2$, there exists a rectangle $r'_2 \subseteq P_2$ that contains $r_2$ and whose boundary contains the top and right sides of $r_2$ only. See Figure 10(a). Then $f_1$ has the horizontal edge of $P$ incident to $a_1$ as edge($f_1$).

Likewise, there exist rectangles $r'_3, r'_4 \subseteq P_2$ such that $r_2 \subset r'_3$ and $r_3 \subset r'_4$, and therefore face $f_2$ incident to $uv, rf_2$ and $rf_3$ has the horizontal edge of $P$ incident to $a_2$ as edge($f_2$). See Figure 10(b).

Finally, there exist rectangles $r'_1, r'_3 \subseteq P_3$ such that $r_1 \subset r'_1$ and $r_3 \subset r'_3$, and therefore face $f_3$ incident to $rf_1, rf_2$ and $rf_3$ has the vertical edge of $P$ incident to $a_3$ as edge($f_3$). See Figure 10(c).

Clearly, face $f_i$ is monotone with respect to edge($f_i$) for all $i \in \{1, 2, 3\}$.

Figure 10: A half-open valley $uv$ can be constructed by taking upper envelope of $R^*(P_1) \cup R^*(P_2) \cup R^*(P_3)$. (a) Face $f_1$ has the horizontal edge incident to $a_1$ as edge($f_1$), (b) face $f_2$ has the horizontal edge incident to $a_2$ as edge($f_2$), and (c) face $f_3$ has the vertical edge incident to $a_3$ as edge($f_3$).

Assume that three reflex vertices of a candidate triple are oriented and placed as depicted in Figure 9. If three reflex vertices $a_1, a_2$ and $a_3$ satisfy the conditions in Lemma 5, we call $(a_1, a_2, a_3)$ a candidate triple of footholds for $uv$, and $\bigcup_{i \in \{1, 2, 3\}} r_i$ the free space of $uv$. 

8
Compatibility Given candidate pairs and triples of footholds for open and half-open valleys, respectively, we need to check whether there is a realistic roof that contains these valleys. In some cases, there is no realistic roof that contains two given valleys because of the geometric constraints of realistic roofs. We say a pair of valleys are compatible if there is a realistic roof that contains them.

We start with a lemma which states the compatibility between two open valleys.

**Lemma 6** Let \((a_1, a_2)\) and \((a'_1, a'_2)\) be two candidate pairs of footholds for open valleys \(uv\) and \(u'v'\), respectively. \((a_1, a_2)\) and \((a'_1, a'_2)\) are compatible if and only if \(\overline{C}_{a_1 a_2} \cap \overline{C}_{a'_1 a'_2} = \emptyset\), where \(\overline{C}_{a_1 a_2} := a_1 \overline{uv} \cup \overline{uv} \cup \overline{v} a_2\) and \(\overline{C}_{a'_1 a'_2} := a'_1 \overline{u'v'} \cup \overline{u'v'} \cup \overline{v'} a'_2\).

There are two cases to consider: compatibility between two half-open valleys, and compatibility between an open valley and a half-open valley.

**Lemma 7** Let \((a_1, a_2, a_3)\) and \((a'_1, a'_2, a'_3)\) be candidate triples of footholds for two half-open valleys \(uv\) and \(u'v'\). Two half-open valleys \(uv\) and \(u'v'\) are compatible if and only if the free space of \(uv\) is contained in one of three rectilinear subpolygons of \(P\) defined by \((a'_1, a'_2, a'_3)\), and the free space of \(u'v'\) is completely contained in one of three rectilinear subpolygons of \(P\) defined by \((a_1, a_2, a_3)\).

**Proof.** Let \(P_i\) and \(P'_i\), for \(i \in \{1, 2, 3\}\), be the rectilinear subpolygons of \(P\) defined by \((a_1, a_2, a_3)\) and \((a'_1, a'_2, a'_3)\), respectively. We can assume that all \(a'_i\) are contained in \(\partial P_i\), for some \(i \in \{1, 2, 3\}\); otherwise, a roof edge associated with \(uv\) and a roof edge associated with \(u'v'\) cross, for which there is no realistic roof containing \(uv\) and \(u'v'\). This also implies that all \(a_i\) are contained in \(\partial P'_i\) for some \(i \in \{1, 2, 3\}\). Consider the case that all \(a_i\) are contained in \(\partial P'_1\), and therefore all \(a'_i\) are contained in \(\partial P_1\). Assume to the contrary that the free space of \(uv\) is not contained in any of \(P'_1, P'_2\) and \(P'_3\), as depicted in Figure 11(a). This implies that \(v_1\) intersects \(\partial P'_1\) and \(y(a_1) - y(a'_1) < z(a_3) - z(a_1)\). Let \(s\) and \(s'\) denote the two peak points of \(uv\) and \(u'v'\), respectively. Let \(p\) be the point \(h \cap (a'_i s' \cup s' u' \cup u'v')\), where \(h\) is the plane through \(s\) and parallel to the \(yz\)-plane. Since \(y(s) < (y(a_1) + y(a'_1))/2\), we have \(y(s) - y(p) < z(s) - z(p)\) and therefore the portion of \(R \cap h\) from \(s\) to \(p\) must have an edge of slope larger than 1, which is not allowed in a realistic roof. The remaining two cases that all \(a_i\) are contained in either \(\partial P'_2\) or \(\partial P'_3\) can also be shown to make \(uv\) and \(u'v'\) not compatible by using a similar argument.

Suppose now that the free space of \(uv\) is contained in one of three rectilinear subpolygons of \(P\) defined by \((a'_1, a'_2, a'_3)\), and the free space of \(u'v'\) is completely contained in one of three rectilinear subpolygons of \(P\) defined by \((a_1, a_2, a_3)\). We show how to construct a realistic roof with \(uv\) and \(u'v'\). Without loss of generality, we assume that \(P_i\) contains \(a'_i, a'_2\) and \(a'_3\). Let \(P_{11}, P_{12}\) and \(P_{13}\) denote the rectilinear subpolygons of \(P_i\) defined by \((a'_1, a'_2, a'_3)\). Now we have five rectilinear subpolygons \(P_{11}, P_{12}, P_{13}, P_2\) and \(P_3\) of \(P\). By taking the upper envelope of the roofs \(R'(P_{11}), R'(P_{12}), R'(P_{13}), R'(P_2)\) and \(R'(P_3)\), we can get a realistic roof which contains \(uv\) and \(u'v'\).

**Lemma 8** Let \(uv\) be a half-open valley and let \((a'_1, a'_2)\) be a candidate pair of footholds for an open valley \(u'v'\). Two valleys \(uv\) and \(u'v'\) are compatible if and only if the smallest axis-aligned rectangle containing \(a'_1\) and \(a'_2\) does not cross the free space of \(uv\) properly.

**Proof.** Let \(P_i\), for \(i \in \{1, 2, 3\}\), be the rectilinear subpolygons of \(P\) defined by the triple \((a_1, a_2, a_3)\) of footholds of \(uv\). We can assume that \(a'_1\) and \(a'_2\) are contained in \(\partial P_i\) for some \(i \in \{1, 2, 3\}\); otherwise, a roof edge associated with \(uv\) and a roof edge associated with \(u'v'\) cross, for which there is no realistic roof containing \(uv\) and \(u'v'\). Let \(B\) denote the smallest axis-aligned rectangle containing \(a'_1\) and \(a'_2\). If \(a'_1\) and \(a'_2\) are contained in \(\partial P_2\) or \(\partial P_3\), then \(B\) does not cross the free space of \(uv\) properly.

Suppose that \(a'_1\) and \(a'_2\) are contained in \(\partial P_1\) and \(B\) crosses the free space of \(uv\) properly, as depicted in Figure 11(b). Let \(p\) be the point \(h \cap (a'_1 u' \cup u'v' \cup v'a'_2)\), where \(h\) is the plane through \(s\) and parallel to the \(yz\)-plane. Since \(y(s) < (y(a_1) + y(a'_1))/2\), we have \(y(s) - y(p) < z(s) - z(p)\) and therefore the portion of \(R \cap h\) from \(s\) to \(p\) must have an edge of slope larger than 1, which is not allowed in a realistic roof.

Suppose now that \(B\) does not cross the free space of \(uv\) properly. We show how to construct a realistic roof with \(uv\) and \(u'v'\). Ahn et al. [1] showed how to construct a realistic roof \(R\) over \(P\) with a candidate...
Figure 11: (a) The free space of $uv$ (gray) crosses $\partial P'_1$. Then we have $y(s) - y(p) < z(s) - z(p)$, for which we cannot construct a realistic roof. (b) The free space of $uv$ crosses $B$ properly. Then we have $y(s) - y(p) < z(s) - z(p)$, for which we cannot construct a realistic roof.

Let $V$ be a set of candidate pairs of footholds and candidate triples of footholds. If every pair of elements of $V$ satisfies Lemma 6 or Lemma 7 or Lemma 8, we can find a unique realistic roof $R$ whose valleys correspond to $V$. Also, we call such $V$ a compatible valley set of $P$. We conclude this section with the following theorem.

**Theorem 1** Let $P$ be a rectilinear polygon with $n$ vertices and $V$ be a compatible valley set of $k$ candidate pairs of footholds and $l$ candidate triples of footholds with respect to $P$. Then there exists a unique realistic roof $R$ whose valleys correspond to $V$. In addition, there exist $k + 2l + 1$ rectilinear subpolygons $P_1, \ldots, P_{k+2l+1}$ of $P$ such that

1. $\bigcup_{i=1}^{k+2l+1} P_i = P$.
2. $R$ coincides with the upper envelope of $R^*(P_i)$'s, for all $i = 1, \ldots, k + 2l + 1$.

### 5 The Number of Realistic Roofs

We give an upper bound of the number of possible realistic roofs over $P$ in terms of $n$. For this, we need a few technical lemmas.

**Lemma 9** Let $(a_1, a_2, a_3)$ be a candidate triple of footholds for a half-open valley, where $a_1$ and $a_2$ have opposite orientations. Then $(a_1, a_2)$ is also a candidate pair of footholds.

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Lemma 10 Let \((a_1, a_2, a_3)\) be a candidate triple of footholds for a half-open valley \(uv\), where \(a_1\) and \(a_2\) have opposite orientations. If a candidate pair \((a_1, a_5)\) of footholds for an open valley is compatible with \((a_1, a_2, a_3)\), then there is no half-open valley with footholds \((a_3, a_4, a_5)\).

Proof. Without loss of generality, assume that the three reflex vertices \(a_1, a_2, a_3\) and the valley \(uv\) are oriented and placed as in Figure 12(a). By Lemma 8, both \(a_4\) and \(a_5\) must be contained in one of three rectilinear subpolygons \(P_1, P_2\) and \(P_3\) of \(P\) defined by \(uv\). Assume to the contrary that \((a_3, a_4, a_5)\) is a candidate triple of footholds for a half-open valley \(u'v'\). There are four cases for \((a_4, a_5)\) as follows.

If \(a_4, a_5 \in \partial P_3\), there is only one possible configuration as depicted in Figure 12(b). By some careful case analysis, we have \(d_x(3, 5) > d_x(3, 4) > d_y(3, 4)\), which makes \(a_4\) be contained in the interior of the free space of \(u'v'\). In case that \(a_4, a_5 \in \partial P_2\), there is no possible configuration. Finally, consider the case that \(a_4, a_5 \in \partial P_1\). There are two possible configurations. When \(x(a_5) < x(a_4) < x(a_1)\) as depicted in Figure 12(c), we have \(d_x(3, 5) > d_x(3, 1) > d_y(3, 1)\), which makes \(a_1\) be contained in the interior of the free space of \(u'v'\).

When \(x(a_5) < x(a_4) < x(a_1)\) as depicted in Figure 12(d), we have \(d_y(3, 4) > d_x(3, 1) > d_y(3, 1)\), which again makes \(a_4\) be contained in the interior of the free space of \(u'v'\).

Figure 12: Illustration of the proof of Lemma 10. Gray regions are free spaces.

Based on the two previous lemmas, we give an upper bound on the number of realistic roofs over \(P\).

Theorem 2 Let \(P\) be a rectilinear polygon with \(n\) vertices. There are at most \(1.3211^m\left(\frac{m}{2}\right)\) distinct realistic roofs over \(P\), where \(m = \frac{n-4}{2}\).

Proof. Let \(R\) be a realistic roof over \(P\) with a half-open valley \(uv\). By Lemma 9 we can get an open valley \(u'v'\) induced by two footholds of \(uv\) that have opposite orientations. Therefore, we can get a new realistic roof by replacing \(uv\) with \(u'v'\). By repeating this process, we can get a realistic roof \(R'\) which does not contain any half-open valleys. It means that for any realistic roof \(R\) over \(P\), there exists a unique realistic roof \(R'\) which has no half-open valleys. We can get the number of distinct realistic roofs over \(P\) with two steps: counting the number of realistic roofs \(R'\) over \(P\) which has no half-open valleys and counting the number of realistic roofs \(R\) which can be transformed to each \(R'\) by replacing its half-open valleys with open valleys.

Ahn et al. [1] gave an upper bound on the number of realistic roofs \(R'\) over \(P\) which have no half-open valleys, which is \(\binom{m}{\frac{m}{2}}\), where \(m = \frac{n-4}{2}\). We calculate the number of realistic roofs \(R\) over \(P\) corresponding to each \(R'\). Suppose that \(R'\) contains \(k\) open valleys, \(u_1v_1, u_2v_2, \ldots, u_kv_k\). \(P\) has \(m = 2k\) reflex vertices that are not used as footholds of these open valleys. Let us call these reflex vertices free vertices of \(R'\). By Lemma 10, each free vertex can make a half-open valley with at most one open valley. Let \(x_i, 1 \leq i \leq k\), be the number of free vertices of \(R'\) that can make a half-open valley with \(u_iv_i\). Then the number of realistic
roofs that can be transformed to $R'$ is at most $(x_1+1)(x_2+1)\cdots(x_k+1)$, where $x_1+x_2+\ldots+x_k \leq m-2k$.

From the inequality of arithmetic and geometric means, we can get

$$
(x_1+1)(x_2+1)\cdots(x_k+1) \leq \left(\frac{x_1+x_2+\ldots+x_k+k}{k}\right)^k
\leq \left(\frac{m-k}{k}\right)^k
= \left(\frac{(m-1)}{m}\right)^m.
$$

For a positive real number $x$, we have $\sup\{(x-1)^{\frac{1}{x}}\} \approx 1.3210998$, so $\left(\frac{m-1}{m}\right)^m < 1.3211^m$. Therefore, we can get at most $1.3211^m$ different realistic roofs over $P$ corresponding to each $R'$, and the total number of distinct realistic roofs over $P$ is at most $1.3211^m\left(\frac{m}{m^2}\right)$. \[\square\]

In the case of an orthogonally convex rectilinear polygon $P$, we can get a better upper bound on the number of realistic roofs over $P$. An orthogonally convex rectilinear polygon is a simple rectilinear polygon such that for any line segment parallel to any of the coordinate axes connecting two points lying within the polygon lies completely within the polygon. The boundary of an orthogonally convex rectilinear polygon consists of four staircases \[\square\]. See Figure 13.

From Lemma 4, a half-open valley $uv$ has three footholds $a_i, a_j$, and $a_k$, which are reflex vertices of $P$ in mutually different orientations, and therefore each of which is contained in a different staircase. Also from Lemma 4, a realistic roof of $P$ containing $uv$ can contain only one additional half-open valley $u'v'$ because only one chain of $\partial P \setminus \{a_i, a_j, a_k\}$ can have three reflex vertices of mutually different orientations. Therefore, all realistic roofs over an orthogonally convex rectilinear polygon can have at most two half-open valleys as shown in Figure 13.

We give an upper bound on the number of realistic roofs over $P$ as we did in the proof of Theorem 2. Let $R'$ be a realistic roof over $P$ which has no half-open valleys and let $k$ denote the number of open valleys $u_1v_1, u_2v_2, \ldots, u_kv_k$ in $R'$. Let $x_i$ denote the number of free vertices of $P$ which can induce a half-open valley with $u_iv_i$. The number of realistic roofs that can be transformed to $R'$ is at most $\sum_{i,j} x_ix_j \leq \binom{k}{2}m^2 \leq m^3$.

Therefore, the number of distinct realistic roofs over $P$ is at most $m^3\left(\frac{m}{m^2}\right)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13.png}
\caption{An orthogonally convex rectilinear polygon $P$ with two half-open valleys $uv$ and $u'v'$}
\end{figure}

**Theorem 3** Let $P$ be an orthogonally convex rectilinear polygon with $n$ vertices. There are at most $m^3\left(\frac{m}{m^2}\right)$ distinct realistic roofs over $P$, where $m = \frac{n-4}{2}$.

### 6 Algorithm

In this section, we will present an algorithm that generates all possible realistic roofs over a given rectilinear polygon $P$. Ahn et al. \[1\] suggested an efficient algorithm that generates all realistic roofs which do not
contain half-open valleys. Let \textit{GenerateOpenValleys} denote the algorithm. \textit{GenerateOpenValleys} spends \(O(n^4)\) time in preprocessing and generates realistic roofs one by one in \(O(1)\) time each. Our algorithm also spends \(O(n^4)\) time in preprocessing: \(P\) has \(O(n^3)\) triples and \(O(n^2)\) pairs of reflex vertices, and checking whether each triple and pair is a candidate triple or candidate pair takes \(O(n)\) time. And then, we create an empty list \(L_{uv}\) of reflex vertices for each candidate pair of \(uv\) and add a reflex vertex \(a_i\) to \(L_{uv}\) if \(a_i\) and the footholds of \(uv\) form a candidate triple.

Our algorithm works as follows. It runs \textit{GenerateOpenValleys} and gets a realistic roof \(R\) with \(k\) open valleys \(u_1v_1, \ldots, u_kv_k\). A pair \((a_i, a_i')\) of footholds corresponding to \(u_iv_i\), \(1 \leq i \leq k\), has a list \(L_{u_iv_i}\) of reflex vertices. Our algorithm either chooses a reflex vertex \(w_i\) from \(L_{u_iv_i}\) or not. Let \(V_O\) denote the set of pairs of footholds for which no reflex vertex is chosen, and let \(V_H\) denote the set of triples \((a_i, a_i', w_i)\) such that a reflex vertex \(w_i\) is chosen for \((a_i, a_i')\). If no reflex vertex is chosen for any pair \((a_i, a_i')\) of footholds, that is, \(V_H = \emptyset\), then the realistic roof with open valleys of \(V\) is exactly \(R\). Otherwise, our algorithm checks whether every pair of valleys in \(V_O \cup V_H\) is compatible as follows. Suppose that we have already checked the compatibility of pairs of valleys in \(V_O \cup V_H\) and let \(N_i\) denote the number of valleys in \((V_O \cup V_H) \setminus \{(a_i, a_i', w_i)\}\) incompatible with \((a_i, a_i', w_i)\).

When we replace \(w_i\) with another reflex vertex \(w_i'\) in \(L_{u_iv_i}\), we compute the compatibility between \((a_i, a_i', w_i')\) and each valley in \((V_O \cup V_H) \setminus \{(a_i, a_i', w_i)\}\) only and update \(N_i\). This can be done in \(O(k)\) time.

If \(\sum_{i=1}^{k} N_i = 0\), every pair of valleys in \(V_O \cup V_H\) is compatible, and therefore there is a roof with valleys of \(V_O \cup V_H\). Therefore, our algorithm finds all realistic roofs correspond to \(P\) in \(O(m1.3211^m)\) time.

\textbf{Theorem 4} Given a rectilinear polygon \(P\) with \(n\) vertices, \(m\) of which are reflex vertices, after \(O(n^4)\)-time preprocessing, all the compatible sets of \(P\) can be enumerated in \(O(m1.3211^m\binom{m}{\frac{m}{2}})\) time.

\textbf{References}


