Guarding Points on a Terrain by Watchtowers *

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Abstract

We study the problem of guarding points on an x-monotone polygonal chain, called a terrain, using k watchtowers. A watchtower is a vertical segment whose bottom endpoint lies on the terrain. A point on the terrain is visible from a watchtower if the line segment connecting the point and the top endpoint of the watchtower does not cross the terrain. Given a sequence of point sites lying on a terrain, we aim to partition the sequence into k contiguous subsequences and place k watchtowers on the terrain such that every point site in a subsequence is visible from the same watchtower and the maximum length of the watchtowers is minimized. We present efficient algorithms for two variants of the problem.

1 Introduction

A *terrain* is a graph of a piecewise linear function $f : A \subset \mathbb{R} \to \mathbb{R}$ that assigns a height $f(p)$ to every point p in the domain A of the terrain. In other words, a terrain is an x-monotone polygonal chain in the plane. A *watchtower* is a vertical segment whose bottom endpoint lies on the terrain. A point on the terrain is *visible* from a watchtower if the line segment connecting the point and the top endpoint of the watchtower does not cross the terrain. If a point is visible from a watchtower, we say that the point is *guarded* by the watchtower. We say that a set of points is guarded by a watchtower if every point in the set is guarded by the watchtower.

In this paper, we study the following problem of guarding point sites on a terrain using k watchtowers: Given a sequence of point sites on a terrain, partition it into k subsequences and place k watchtowers on the terrain such that every point site in a subsequence is guarded by the same watchtower and the maximum length of the watchtowers is minimized. We call it the *contiguous k-watchtower problem* for point sites on a terrain. We also consider the problem with an additional condition on the placement of watchtowers: a watchtower guarding a subsequence of point sites must be placed in the x-range $x_{\min} \leqslant x_w \leqslant x_{\max}$ of the point sites in the subsequence, where x_w is the x-coordinate of the watchtower and $x_{\min}$ (resp. $x_{\max}$) is the minimum (resp. maximum) x-coordinates of the point sites in the subsequence. This is the *in-place* version of the contiguous k-watchtower problem for point sites on a terrain. For both problems, we call those k watchtowers satisfying the conditions and minimizing the maximum length the *optimal k* watchtowers. See Figure 1 for an illustration for the problems.

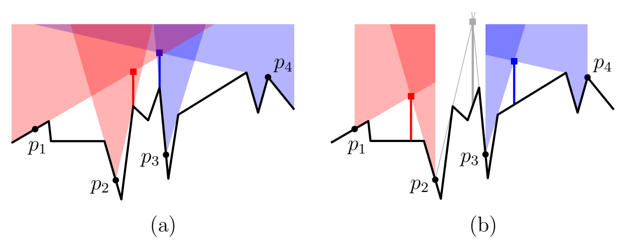

Figure 1: (a) Optimal watchtowers for the contiguous 2-watchtower problem. The red tower guards p_1 and p_2, and the blue tower guards p_3 and p_4. (b) Optimal watchtowers for the in-place version. The red watchtower guards p_1 and p_2, and it is placed in the x-range of p_1 and p_2. The blue watchtower guards p_3 and p_4, and it is placed in the x-range of p_3 and p_4. To guard point sites including both p_2 and p_3 using one watchtower, the watchtower must be at least as long as the gray watchtower.

The k-watchtower problems we consider have applications in several domains, including geographic information system, communication tower locations, and military surveillance [4].

1.1 Related works

A fair amount of work has been done on minimizing the number of guards in various settings. The art gallery problem [10] asks for the minimum number of point

*This research was supported by the Institute of Information & communications Technology Planning & Evaluation(IITP) grant funded by the Korea government(MSIT) (No. 2017-0-00905, Software Star Lab (Optimal Data Structure and Algorithmic Applications in Dynamic Geometric Environment)) and (RS-2019-II191906, Artificial Intelligence Graduate School Program(POSTECH)). This work was also supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) (RS-2023-00219980).

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guards that together guard the whole art gallery, represented by a simple polygon. The art gallery problem was first posed by Klee in 1973 [10]. Chvátal and Fisk [5, 8] gave an upper bound $\lfloor n/3 \rfloor$ on the minimum number of point guards for a simple polygon with n vertices.

The terrain guarding problem [9] asks for the minimum number of point guards lying on the terrain that together guard the terrain. Cole and Sharir [6] showed that finding the minimum number of guards for a polyhedral terrain in 3-dimensional space is NP-complete. Later, Chen et al. [3] showed that the same problem for a terrain in 2-dimensional space is also NP-complete.

The k-watchtower problem for a terrain with n vertices in 2-dimensional space is to minimize the maximum length of k watchtowers that together guard the whole terrain. The 2-watchtower problem was first studied by Bespamyatnikh et al. [2]. They presented an $O(n^3 \log^2 n)$-time algorithm for the variant, called the discrete version, in which every watchtower must be placed at a vertex of the terrain. They also gave an $O(n^4 \log^2 n)$-time algorithm for the continuous version in which the two watchtowers can be placed anywhere in the terrain. Agarwal et al. [1] improved the results by an $O(n^2 \log^4 n)$-time algorithm for the discrete version and by an $O(n^3 \alpha(n) \log^3 n)$-time algorithm for the continuous version.

There are also a few results for the k-watchtower problem for a 2-dimensional terrain with n vertices in 3-dimensional space. Agarwal et al. [1] presented an $O(n^{11/3}\text{polylog}(n))$-time algorithm for the discrete version of the 2-watchtower problem. Recently, Tripathi et al. [12] gave an algorithm for the discrete version of the k-watchtower problem that runs in $O(n^{k+3}k^2\alpha^2(n)\log^2 n + n^7\alpha^3(n)\log n)$ time.

To the best of our knowledge, little is known about guarding a finite set of input points lying on a terrain, not the whole terrain, except the one by Agarwal et al. [1]. They considered the 2-watchtower problem for guarding a finite set of m point sites on a terrain with n vertices in 2-dimensional space where every point site must be guarded by at least one of the two watchtowers. The watchtowers can be placed anywhere in the terrain. They presented an $O(mn \log^4 n)$-time algorithm for the problem. One may wonder if this algorithm extends to the k-watchtower problem for $k \geqslant 3$. It seems to us that it does, but the running time becomes exponential in k for m point sites lying on a terrain with n vertices.

1.1.1 Our results.

We consider the contiguous k-watchtower problem and the in-place contiguous k-watchtower problem for m point sites lying on a terrain with n vertices in the plane. For ease of the description, we may call the in-place contiguous k-watchtower problem the in-place k-watchtower problem. If $k \geqslant m$ (resp. $k \geqslant n$), we place one watchtower with zero length on every point site (resp. on every vertex of the terrain). Considering the cost of watchtowers, it is desirable to use a small number of watchtowers for point sites. Therefore, we assume that $k \ll \min\{n, m\}$.

For $k = 1$, we present an algorithm with running time $O(m + n)$ for both problems. Observe that the running time is linear to the complexity of the input. This is an improvement upon the previously best algorithm with running time $O(mn)$ [1].

For the contiguous k-watchtower problem, the watchtowers can be placed anywhere in the terrain. We show a monotonicity on the minimum length of a watchtower, and present an $O((m + n) \log m)$-time algorithm for $k = 2$. For $k \geqslant 3$, we can solve the problem in $O(k(n + m) \log^{\lceil \log_2 k \rceil} m)$ time. Our algorithm runs in $O((m + n) \log^{\lceil \log_2 k \rceil} m)$ time for any fixed k.

For the in-place k-watchtower problem, a watchtower guarding a contiguous subsequence of point sites must be placed in the x-range of the subsequence. We observe that the monotonicity shown for the contiguous k-watchtower problem does not hold for this problem. We present an $O((m + n) \log(m + n))$-time algorithm for $k = 2$ and an $O(km^2 + (mn + m^2) \log(m + n))$-time algorithm for $k \geqslant 3$. Our algorithm runs in $O((mn + m^2) \log(m + n))$ time for any fixed $k \geqslant 3$.

1.1.2 Sketch of our algorithms.

We devise an efficient algorithm for the contiguous k-watchtower problem for $k = 1$ that runs in $O(m + n)$ time. The *visibility region* of a point site is the set of points visible from the point site. To find an optimal watchtower, we need to compute the intersection of the visibility regions of point sites. The previous algorithm takes $O(mn)$ time in computing visibility regions of point sites and their intersection [1]. To do this efficiently, we define a region $W(p, q)$ for a pair of point sites (p, q) such that $W(p, q)$ contains the intersection of the visibility regions of p and q. We show that the intersection of visibility regions of all point sites can be computed in $O(m + n)$ time using the intersection of $W(p, q)$'s for all pairs of point sites (p, q). From this, we can compute an optimal watchtower for m point sites lying on a terrain with n vertices in $O(m + n)$ time.

For $k \geqslant 2$, we show a monotonicity stating that the length of an optimal watchtower for a subsequence P_1 of point sites is at least the length of an optimal watchtower for any subsequence of P_1. Based on the monotonicity, our algorithm for the contiguous k-watchtower problem uses binary search to find an optimal partition of the point site set that minimizes the maximum length of the watchtowers. In each step of the binary search, we partition the point site sequence into two contiguous subsequences. Then, we compute the optimal length of the watchtowers for each subsequence using half of the watchtowers. If the optimal length of the watchtowers for the left subsequence is larger than the right subsequence's, then we find an optimal partition index in the left half of indices of the point site set. When the number of the watchtower is one, we can compute the optimal length of the half of the watchtowers in $O(m+n)$ time by using the algorithm for one watchtower.

In the in-place k-watchtower problem, the monotonicity used for our algorithm for the contiguous k-watchtower problem does not hold. So we consider every possible partition of the sequence into k contiguous subsequences. For $k=2$, there are $O(m)$ different partitions. A naïve approach is to compute the optimal tower-length for every partition in $O(m^2+mn)$ total time by applying the algorithm for the contiguous 1-watchtower problem. We compute optimal watchtowers efficiently as follows. For every prefix of the input sequence of point sites, we compute the intersection of $W(p,q)$'s for every pair of point sites (p,q) in the prefix. We compute those intersections incrementally in the length of the prefixes in $O((m+n)\log(m+n))$ total time. Using those intersections, we can compute optimal two watchtowers in $O((m+n)\log(m+n))$ time.

For $k \geqslant 3$, a naïve approach is to consider $O(m^{k-1})$ different partitions, compute their optimal tower-lengths, and then return the minimum one among them. To compute optimal k watchtowers efficiently, we compute the minimum length of one watchtower for every contiguous subsequence incrementally in $O((m^2+mn)\log(m+n))$ total time in the preprocessing. Then we find an optimal partition by dynamic programming that has $O(km^2)$ subproblems.

Most proofs are omitted and they will be given in a full version.

2 Preliminaries

For a point p in the plane, we use $x(p)$ and $y(p)$ to denote the x- and y-coordinates of p. For two distinct points p and q in the plane, let pq denote the line segment connecting p and q, and let $\overline{pq}$ denote the line passing through both p and q. For a nonvertical line L, we use L^+ to denote the set of points in $\mathbb{R}^2$ that lie on or above L, and L^- to denote the set of points in $\mathbb{R}^2$ that lie on or below L.

A region A is *x-monotone* if for every line L perpendicular to the x-axis, $A \cap L$ is connected. A region A is *unbounded vertically upwards* if any vertically upward ray emanating from a point in A is contained in A. A polygonal chain B is *x-monotone* if for every line L perpendicular to the x-axis, either $B \cap L = \emptyset$ or it is a point. We use $T = \langle v_1, \ldots, v_n \rangle$, a sequence of vertices with $x(v_i) < x(v_j)$ for any $1 \leqslant i < j \leqslant n$, to denote an x-monotone polygonal chain which we call a *terrain* in 2-dimensional space. Without loss of generality, we assume $n \geqslant 2$. For any two points $p, q \in T$ with $x(p) \leqslant x(q)$, let $T(p,q)$ denote the subchain of T from p to q, and let $T^+(p,q)$ denote the set of points $z \in \mathbb{R}^2$ such that $x(p) \leqslant x(z) \leqslant x(q)$ and $y(z) \geqslant y(z')$, where z' is a point in T with $x(z) = x(z')$. We simply use T^+ to denote $T^+(v_1, v_n)$. We denote by $P = \langle p_1, \ldots, p_m \rangle$ a sequence of m point sites lying on T such that $x(p_i) < x(p_j)$ for $1 \leqslant i < j \leqslant m$. We denote by $P(i,j)$ the contiguous subsequence $\langle p_i, \ldots, p_j \rangle$ of P for $1 \leqslant i < j \leqslant m$. For ease of description, we assume that $m \geqslant 2$, and let $p_0 = v_1$ and $p_{m+1} = v_n$. We use $T(i,j)$ to denote $T(p_i, p_j)$, and $T^+(i,j)$ to denote $T^+(p_i, p_j)$.

A point $p \in \mathbb{R}^2$ is *visible* from a point $q \in \mathbb{R}^2$ if and only if pq is contained in T^+. For a point $q \in T$, let $V(q)$ denote the *visibility region* of q, which consists of the points in T^+ visible from q. For a point site $p_i \in P$, we use $V(i)$ to denote $V(p_i)$. Observe that $V(i)$ is connected and unbounded vertically upwards. Let $\mathbb{V}(i,j) = \bigcap_{i \leqslant \ell \leqslant j} V(p_\ell)$. The following observation is straightforward.

Observation 1 *The point sites in $P(i,j)$ are visible from a watchtower if and only if the top endpoint of the watchtower is contained in $\mathbb{V}(i,j)$.*

For any two real values a, b with $a \leqslant b$, we use $S(a,b)$ to denote the vertical slab between the lines $x = a$ and $x = b$. In other words, it is the set of points $z \in \mathbb{R}^2$ such that $a \leqslant x(z) \leqslant b$. For any two points $p, q \in \mathbb{R}^2$ with $x(p) \leqslant x(q)$, we abuse the notation so that $S(p,q)$ denotes $S(x(p), x(q))$. We use $S(i,j)$ to denote $S(p_i, p_j)$. For a set $A \subset \mathbb{R}^2$, we use $S(A)$ to denote the smallest vertical slab containing A.

For any two sets A and B of points, let $d_y(A,B)$ denote the minimum vertical distance between A and B, that is, $d_y(A,B) = \min_{p_A \in A, p_B \in B} |y(p_A) - y(p_B)|$ subject to $x(p_A) = x(p_B)$. If there are no two points $p_A \in A$ and $p_B \in B$ with $x(p_A) = x(p_B)$, we set $d_y(A,B) = \infty$. We say that A lies *left* to B if the rightmost point p of A and the leftmost point of q of B satisfy $x(p) \leqslant x(q)$.

3 Contiguous k watchtowers

In this section, we present an $O(k(n+m)\log^{\lceil \log_2 k \rceil} m)$-time algorithm for the contiguous k-watchtower problem for point sites P on a terrain T. In Section 3.1, we present an $O(m+n)$-time algorithm for computing an optimal watchtower for $P = \langle p_1, \ldots, p_m \rangle$. We use the algorithm for one watchtower together with binary search in computing the optimal k watchtowers for $k \geqslant 2$ in Sections 3.2 and 3.3. For any constant k, the algorithm runs in near-linear time: $O((m+n)\log m)$ time for $k=2$, and $O((m+n)\log^{\lceil \log_2 k \rceil} m)$ time for any fixed k.

3.1 An optimal watchtower for a site sequence

We consider the problem of placing a shortest watchtower that guards all point sites of P. Let $F(1,m)$ denote the minimum length of a watchtower that guards all point sites in P. By Observation 1, any watchtower guarding point sites in P must have its top endpoint contained in $\mathbb{V}(1,m)$. Thus, $F(1,m) = d_y(T, \mathbb{V}(1,m))$.

A straightforward way to compute an optimal watchtower for the sequence is to compute $V(\ell)$ for all $\ell = 1,\ldots,m$, compute their intersection $\mathbb{V}(1,m)$, and then compute $F(1,m)$. Observe that it already takes $O(mn)$ time for computing $V(\ell)$ for all $\ell = 1,\ldots,m$ [11].

We show how to compute $\mathbb{V}(1,m) = \bigcap_{1\leqslant\ell\leqslant m} V(\ell)$ efficiently, in $O(m+n)$ time. Before showing this, we need to define a region $R(1,m)$ for $P(1,m)$. Let L be line $\overline{p_1p_m}$ if p_m is visible from p_1. If p_m is not visible from p_1, let L be line $\overline{uv}$, where uv is the edge of $V(1)$ with $x(u) < x(p_m) \leqslant x(v)$. If p_m lies on a vertex of T, let $R(1,m)$ be the set of points $z \in L^+$ satisfying $x(z) \geqslant x(p_m)$. If p_m is contained in the interior of an edge e of T, let $R(1,m)$ be the set of points $z \in L^+ \cap \bar{e}^+$ satisfying $x(z) \geqslant x(p_m)$. See Figure 2 for an illustration for four possible cases. We define the region $R(m,1)$ symmetrically.

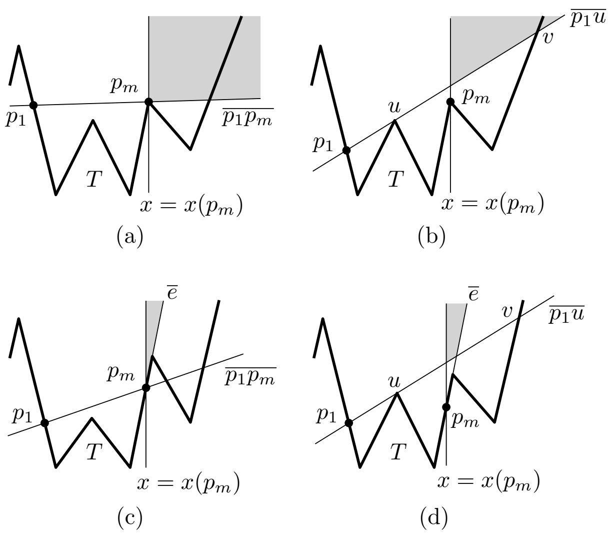

Figure 2: $R(1,m)$ in gray region. (a) p_m lying on a vertex of T and visible from p_1. (b) p_m lying on a vertex of T and not visible from p_1. (c) p_m lying in the interior of an edge e of T and visible from p_1. (d) p_m lying in the interior of an edge e of T and not visible from p_1.

By definition, $R(1,m)$ is the intersection of two or three closed half-planes. Thus, $R(1,m)$ is convex. Moreover, it is unbounded vertically upwards.

Combining $R(1,m), R(m,1)$, and $V(1) \cap V(m)$ restricted to $S(1,m)$, we define $W(1,m)$ as follows. See Figure 3 for an illustration.

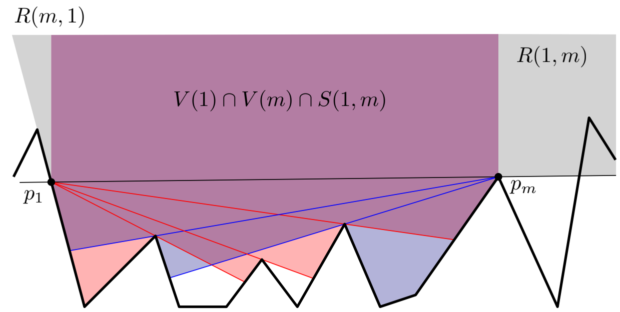

Figure 3: The purple region is $V(1) \cap V(m) \cap S(1,m)$. $W(1,m)$ is the union of the purple region and the right gray region from $R(1,m)$ and the left gray region from $R(m,1)$.

$$W(1,m) = R(1,m) \cup R(m,1) \cup \big(V(1) \cap V(m) \cap S(1,m)\big).$$

By definition, $W(1,m)$ is connected and unbounded vertically upwards.

Observation 2 *The followings hold by the definition of* $W(1,m)$.

(a) $W(1,m) \cap S(1,m) = V(1) \cap V(m) \cap S(1,m)$.
(b) $W(1,m) \cap S(0,1) = R(m,1) \cap S(0,1)$.
(c) $W(1,m) \cap S(m,m+1) = R(1,m) \cap S(m,m+1)$.
(d) *For* $q \in \{p_1, p_m\}$, $y(q) \leqslant y(z)$ *for all* $z \in W(1,m)$ *with* $x(z) = x(q)$.

Based on Observation 2, we can compute $W(1,m)$ efficiently.

Lemma 1 *We can compute* $W(1,m)$ *in time linear to the complexity of* $T(1,m)$.

By Lemma 1, $W(\ell,\ell+1)$ can be computed in time linear to the complexity of $T(\ell,\ell+1)$. Thus, we can compute $W(\ell,\ell+1)$ for all $\ell = 1,\ldots,m$ in $O(m+n)$ time. We show a few properties useful for computing $\mathbb{V}(1,m)$ efficiently.

Lemma 2 $\mathbb{V}(1,m) = \bigcap_{1<\ell\leqslant m} W(\ell-1,\ell) \cap V(1) \cap V(m)$.

Let $X_1 = \bigcap_{1<\ell\leqslant r} R(\ell-1,\ell)$, $X_2 = \bigcap_{r<\ell\leqslant m} R(\ell,\ell-1)$, $X_3 = V(r) \cap V(r+1) \cap V(1) \cap V(m)$, and $X_4 = S(r,r+1)$. By Lemma 2 and Observation 2(c),

$$\mathbb{V}(1,m) \cap S(r,r+1) = X_1 \cap X_2 \cap X_3 \cap X_4. \qquad (1)$$

We need the following lemma to show that $\mathbb{V}(1,m)$ can be computed in $O(m+n)$ time.

Lemma 3 *We can compute* $\bigcap_{1<\ell\leqslant r} R(\ell-1,\ell) \cap S(r,r+1)$ *for all* $r = 2,\ldots,m$ *in* $O(m+n)$ *time.*

Theorem 4 *We can compute a minimum-length watchtower that guards m point sites lying on an x-monotone polygonal chain with n vertices in $O(m+n)$ time.*

Proof. Note that $F(1,m) = d_y(T, \mathbb{V}(1,m))$. First, we show how to compute $\mathbb{V}(1,m)$ in $O(m+n)$ time. We can get $\mathbb{V}(1,m)$ by gluing $\mathbb{V}(1,m) \cap S(0,2)$, $\mathbb{V}(1,m) \cap S(2,m-1)$, and $\mathbb{V}(1,m) \cap S(m-1,m+1)$.

We compute $\mathbb{V}(1,m) \cap S(r,r+1)$ for all $r = 2,\dots,m-1$ which is defined in Equation 1. By Lemma 3, we can compute $\bigcap_{1<\ell\leqslant r} R(\ell-1,\ell) \cap S(r,r+1)$ for all $r = 2,\dots,m-1$ in $O(m+n)$ time. Their total complexity is $O(m+n)$. Similarly, we can compute $\bigcap_{r<\ell\leqslant m} R(\ell,\ell-1) \cap S(r,r+1)$ for all $r = 2,\dots,m-1$ in $O(m+n)$ time. Their total complexity is $O(m+n)$. By Lemma 1, we can compute $V(r) \cap V(r+1) \cap S(r,r+1)$ for all $r = 2,\dots,m-1$ in $O(m+n)$ time. Their total complexity is $O(m+n)$. Recall that we can compute $V(1)$ and $V(m)$ in $O(n)$ time [11]. Observe that every region that we compute is x-monotone. Thus, we can compute the intersections $\mathbb{V}(1,m) \cap S(r,r+1)$ of those regions for all $r = 2,\dots,m-1$ in time linear to their total complexity $O(m+n)$ by linear scan. Similarly, $\mathbb{V}(1,m) \cap S(0,2)$ and $\mathbb{V}(1,m) \cap S(m-1,m+1)$ can be computed in $O(m+n)$ time.

We glue $\mathbb{V}(1,m) \cap S(0,2)$, $\mathbb{V}(1,m) \cap S(2,m-1)$, and $\mathbb{V}(1,m) \cap S(m-1,m+1)$ together and get $\mathbb{V}(1,m)$. Since the complexity of $\mathbb{V}(1,m)$ is $O(m+n)$, we can compute $F(1,m) = d_y(T, \mathbb{V}(1,m))$ in $O(m+n)$ time by linear scan. We compute the location of an optimal watchtower during the scan. □

3.2 Two watchtowers

We consider the contiguous k-watchtower problem for $k = 2$: Partition P into 2 subsequences and place 2 watchtowers on T such that every point site in a subsequence is guarded by the same watchtower and the maximum length of the watchtowers is minimized.

Recall that $P(i,j)$ denotes the contiguous subsequence $\langle p_i,\dots,p_j\rangle$ of $P = \langle p_1,\dots,p_m\rangle$ for $1 \leqslant i < j \leqslant m$. Let $F(i,j)$ denote the minimum length of a watchtower that guards point sites in $P(i,j)$ lying on T. We have the following lemma stating the monotonicity on $F(i,j)$ obtained by $\mathbb{V}(i',j') \subseteq \mathbb{V}(i,j)$.

Lemma 5 *For indices i', i, j and j' satisfying $1 \leqslant i' \leqslant i \leqslant j \leqslant j' \leqslant m$, $F(i,j) \leqslant F(i',j')$.*

For an index i with $1 \leqslant i < m$, let $F_1(i) = F(1,i)$ and $F_2(i) = F(i+1,m)$. Then the minimum length for two watchtowers is $\min_{1\leqslant i<m}\{\max\{F_1(i),F_2(i)\}\}$. By Lemma 5, $F_1(i)$ increases monotonically and $F_2(i)$ decreases monotonically as i increases from 1 to $m-1$. Therefore, we find the index that achieves the minimum length by binary search. Since P consists of m point sites, the number of binary search steps is $O(\log m)$. By Theorem 4, the comparison in each step can be done in $O(m+n)$ time. In other words, we can compute both $F_1(i)$ and $F_2(i)$ for any index $i = 1,\dots,m-1$ in $O(m+n)$ time. Also, we can compute the location of an optimal watchtower for $P(i,j)$ for any index $1 \leqslant i \leqslant j \leqslant m$ in $O(m+n)$ time by Theorem 4. Therefore, we can compute the optimal two watchtowers in $O((m+n)\log m)$ time.

Theorem 6 *We can compute optimal two watchtowers for the contiguous 2-watchtower problem with m point sites lying on an x-monotone polygonal chain with n vertices in $O((m+n)\log m)$ time.*

3.3 k watchtowers

In this section, we present an $O(k(n+m)\log^{\lceil \log_2 k\rceil} m)$-time algorithm for computing the contiguous k watchtowers of minimum length for $k \geqslant 3$. Roughly speaking, we partition P into two contiguous subsequences and compute the minimum tower-length for one subsequence using $\lfloor k/2 \rfloor$ watchtowers and the minimum tower-length for the other subsequence using $\lceil k/2\rceil$ watchtowers. We repeat this recursively.

For indices $1 \leqslant i \leqslant j \leqslant m$, let $\mathsf{opt}(i,j,k')$ denote the minimum tower-length for $P(i,j)$ using k' watchtowers with $k' \geqslant 1$. Obviously, $\mathsf{opt}(i,j,k') \geqslant \mathsf{opt}(i,j,k'+1)$. Observe that $\mathsf{opt}(i,j,1) = F(i,j)$. For $k' \geqslant 2$, $\mathsf{opt}(i,j,k')$ equals to

$$\min_{i\leqslant \ell<j}\{\max\{\mathsf{opt}(i,\ell,\lfloor k'/2\rfloor), \mathsf{opt}(\ell+1,j,\lceil k'/2\rceil)\}\}.$$

Lemma 7 *$\mathsf{opt}(1,i,k') \leqslant \mathsf{opt}(1,j,k')$ for $1 \leqslant i \leqslant j \leqslant m$ and $k' \geqslant 1$.*

The minimum tower-length for $P(1,m)$ using k watchtowers is $\mathsf{opt}(1,m,k)$. By Lemma 7, we can find an index $\ell = \arg\min_{1\leqslant \ell<m}\max\{\mathsf{opt}(1,\ell,\lfloor k'/2\rfloor), \mathsf{opt}(\ell+1,m,\lceil k'/2\rceil)\}$ by binary search. Therefore, we conclude this section with Theorem 8.

Theorem 8 *We can compute optimal k watchtowers for the contiguous k-watchtower problem with m point sites lying on an x-monotone polygonal chain with n vertices in $O(k(n+m)\log^{\lceil \log_2 k\rceil} m)$ time.*

4 In-place contiguous k watchtowers

In this section, we present algorithms for the in-place k-watchtower problem for P lying on T. In this problem, a watchtower that guards a subsequence $P(i,j)$ must be placed in $T(i,j)$. By the problem definition, no watchtower cannot be placed on $T(0,1) \cup T(m,m+1)$. Thus, for ease of discussion, we assume that p_1 lies on v_1 and p_m lies on v_n.

In Section 4.1, we present an $O((m+n)\log(m+n))$-time algorithm for $k=2$. The algorithm works in incremental fashion in computing an optimal solution using a balanced binary search tree based on the segment tree [7]. In Section 4.2, we present an $O(km^2 + (mn+m^2)\log(m+n))$-time algorithm for $k \geqslant 3$. The algorithm uses dynamic programming in computing an optimal solution, using the $O((m+n)\log(m+n))$-time algorithm for $k=2$ for the base case.

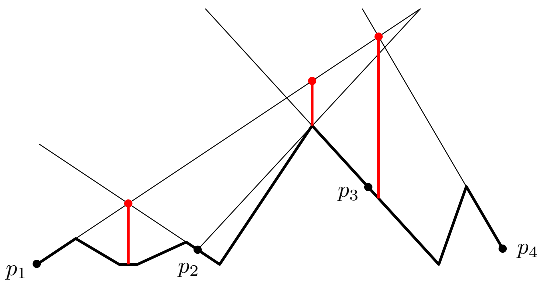

Figure 4: The vertical red line segments, left to right, are the shortest watchtowers for $P(1,2)$, $P(1,3)$, and $P(1,4)$. We have $F_1(2) > F_1(3)$ and $F_1(3) < F_1(4)$.

4.1 Two watchtowers

Let $F(i,j)$ denote the minimum length of one watchtower placed on $T(i,j)$ for $P(i,j)$ with $1 \leqslant i \leqslant j \leqslant m$. Let $F_1(i) = F(1,i)$ and $F_2(i) = F(i+1,m)$. Then our goal is to compute $\min_{1\leqslant i<m}\{\max\{F_1(i), F_2(i)\}\}$.

Observe that the monotonicity in Lemma 5 does not hold for the in-place k-watchtower problem due to the in-place requirement. For two indices i,j with $1 \leqslant i < j \leqslant m$, the watchtower for $P(1,j)$ can be placed anywhere in $T(1,j) = T(1,i) \cup T(i,j)$ while the watchtower for $P(1,i)$ must be placed in $T(1,i)$. So it is possible that $F_1(i) > F_1(j)$. See Figure 4.

We use an incremental algorithm for computing $F_1(i)$ and $F_2(i)$ for all $i = 1,\ldots,m-1$ that runs in $O((m+n)\log(m+n))$ time. Recall that we can compute $W(i-1,i)$ for all $i = 2,\ldots,m$ in $O(m+n)$ time by Lemma 1. Thus, we compute their intersection incrementally.

Let $\mathbb{W}(i) = \bigcap_{1<\ell\leqslant i} W(\ell-1,\ell)$. Recall that $W(\ell-1,\ell)$ is connected and unbounded vertically upwards. Thus, $\mathbb{W}(i)$ is connected and unbounded vertically upwards.

Lemma 9 $\mathbb{V}(1,i) \cap S(1,i) = \mathbb{W}(i) \cap S(1,i)$.

Corollary 10 $d_y(T(1,i), \mathbb{V}(1,i)) = d_y(T(1,i), \mathbb{W}(i))$.

By Observation 1, Lemma 9, and Corollary 10, $F_1(i) = d_y(T(1,i), \mathbb{W}(i))$. Our algorithm starts with trivial base case $F_1(1) = 0$ and computes $F_1(i)$ for all $i = 2,\ldots,m-1$ one by one incrementally.

First, we show that $\mathbb{W}(i)$ for all $i = 2,\ldots,m$ can be computed in $O((m+n)\log(m+n))$ time in total. We can compute $\mathbb{W}(2) = W(1,2)$ in $O(m+n)$ time. We show how to compute $\mathbb{W}(i+1) = \mathbb{W}(i) \cap W(i,i+1)$ from $\mathbb{W}(i)$ efficiently. To do this, we show that the boundary of $W(i,i+1)$ intersects the boundary of $\mathbb{W}(i)$ in $O(|T(i,i+1)|)$ connected components. In specific, each edge of $W(i,i+1)$ intersects the boundary of $\mathbb{W}(i)$ at most twice.

Lemma 11 *We can compute $F_1(i)$ and $F_2(i)$ for all $i = 1,\ldots,m-1$ in $O((m+n)\log(m+n))$ time.*

Recall that the minimum tower-length is $\min_{1\leqslant i<m}\{\max\{F_1(i), F_2(i)\}\}$. By Lemma 11, we can compute $F_1(i)$ and $F_2(i)$ in $O((m+n)\log(m+n))$ time for all $i = 1,\ldots,m-1$. Then, we can find $\min_{1\leqslant i<m}\{\max\{F_1(i), F_2(i)\}\}$ in $O(m)$ time. Recall that we can compute an optimal watchtower that guards $P(i,j)$ in $O(m+n)$ time by Theorem 4. In conclusion, we can compute the minimum tower-length and the locations of the optimal watchtowers in $O((m+n)\log(m+n))$ time.

Theorem 12 *We can compute optimal two watchtowers for the in-place contiguous 2-watchtower problem with m point sites lying on an x-monotone polygonal chain with n vertices in $O((m+n)\log(m+n))$ time.*

4.2 k watchtowers

Now we consider the in-place contiguous k watchtower problem for $k \geqslant 3$. By the definition of the problem, the minimum tower-length is

$$\min_{1\leqslant i_1<\ldots<i_{k-1}<m}\{\max\{F(1,i_1),\ldots,F(i_{k-1}+1,m)\}\}.$$

A naïve approach is to consider all combinations of $k-1$ point sites with indices $1 \leqslant i_1 < \ldots < i_{k-1} < m$ among m point sites, compute their maximum tower-lengths $\max\{F(1,i_1), F(i_1+1,i_2),\ldots,F(i_{k-1}+1,m)\}$, and then return the minimum one among the tower-lengths. This takes $O(m^{k-1}(m+n))$ time.

We can improve the running time using dynamic programming as follows. For an index $1 \leqslant i \leqslant m$, let $\mathsf{opt}(i,k')$ denote the minimum tower-length for the in-place k'-watchtower problem for $P(1,i)$. Then (1) $\mathsf{opt}(i,1) = F(1,i)$, (2) $\mathsf{opt}(i,k') = 0$ if $k' > 1$ and $i \leqslant k'$, and (3) $\mathsf{opt}(i,k') = \min_{1\leqslant j<i}\{\max\{\mathsf{opt}(j,k'-1), F(j+1,i)\}\}$ if $k' > 1$ and $i > k'$.

The optimal length is $\mathsf{opt}(m,k)$ and the number of subproblems is $O(km^2)$. To obtain $\mathsf{opt}(m,k)$, we need to compute $F(i,j)$ for all $1 \leqslant i \leqslant j \leqslant m$. By Theorem 12, for a fixed index $1 \leqslant i \leqslant m$, we can compute $F(i,j)$ for all $i \leqslant j \leqslant m$ in $O((m+n)\log(m+n))$ time. Therefore, we have the following lemma.

Lemma 13 *We can compute $F(i,j)$ for every $1 \leqslant i \leqslant j \leqslant m$ in $O((mn+m^2)\log(m+n))$ time.*

After $O((mn+m^2)\log(m+n))$-time preprocessing by Lemma 13, we can compute the minimum tower-length in $O(km^2)$ time using dynamic programming.

Theorem 14 *We can compute optimal k watchtowers for the in-place contiguous k-watchtower problem with m point sites lying on an x-monotone polygonal chain with n vertices in $O(km^2+(mn+m^2)\log(m+n))$ time.*

We would like to mention that the algorithm presented in this paper also work with little modification and without increasing the running time for minimizing the sum of the tower-lengths for k watchtowers.

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