Universal convex covering problems under affine dihedral group actions*

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Abstract

We consider the smallest-area universal convex H_k-covering of a set of planar objects, which covers every object in the set allowing the group action of the affine dihedral group $H_k = T \rtimes D_k$ generated by the translation T and the dihedral group D_k. The dihedral group D_k is the group of symmetries of a regular polygon generated by the discrete rotation group Z_k and a reflection. We first classify the smallest-area convex H_k-coverings of the set of all unit segments. Then we show that a suitably positioned equilateral triangle of height 1 is a universal convex H_1-covering of the set S_{c} of all closed curves of length 2. We show that no proper closed subset of the covering is a H_1-covering and the covering is a smallest-area triangle H_1-covering of S_{c}. We conjecture that it is the smallest-area convex H_1-covering of S_{c}. We also show that a suitably positioned equilateral triangle $\triangle_\beta$ of height 0.966 is a universal convex H_2-covering of S_{c}. Finally, we give a universal convex H_3-covering of S_{c} whose area is strictly smaller than that of $\triangle_\beta$.

1 Introduction

Given a (possibly infinite) set S of planar objects and a group G of geometric transformations, a universal G-covering K of S is a region such that every object in S can be contained in K by transforming the object with a suitable transformation $g \in G$. Equivalently, every object of S is contained in $g^{-1}K$ for a suitable transformation $g \in G$. That is,

$$\forall \gamma \in S,\ \exists g \in G \text{ such that } \gamma \subseteq g^{-1}K.$$

A G-covering K of S is *minimal* if no proper closed subset of K is a G-covering of S. We denote the group of planar translations by T and that of planar translations and rotations by TR. Mathematically, $TR = T \rtimes R$ is the semidirect product of T and the rotation group (i.e., the two-dimensional special orthogonal group) $R = SO(2, \mathbb{R})$. We denote O for the orthogonal group $O(2, \mathbb{R})$, which is generated by the rotation group R and a reflection (say, with respect to the x-axis). Our group G is a subgroup of $TO = T \rtimes O$, which is the group generated by T and O. The group TO contains every affine linear transformation for which the shapes of geometric objects are invariant. For simplicity, we often call a universal G-covering a G-covering, or a covering if G is known from the context.

The problem of finding a smallest-area covering is a classical problem in mathematics. In the literature, the cases where $G = T$ or $G = TR$ have been widely studied.

The universal covering problem has attracted many mathematicians. Henri Lebesgue (in his letter to J. Pál in 1914) proposed a problem to find the smallest-area convex TR-covering of all objects of unit diameter (see [7, 4, 11] for its history). Soichi Kakeya considered in 1917 the T-covering of the set S_{seg} of all unit line segments (called *needles*) [15]. Precisely, his formulation is to find the smallest-area region in which a unit-length needle can be turned round, but it is equivalent to the T-covering problem if the covering is convex [3]. Fujiwara conjectured that the equilateral triangle of height 1 is the smallest-area convex T-covering of S_{seg}. The conjecture was affirmatively solved by Pál in 1920 [21]. For the nonconvex variant of the Kakeya problem, Besicovitch [5] gave a construction such that the area can be arbitrarily small, and its variants are widely studied with strong influence on several fields of mathematics [9, 24].

Generalizing Pál's result, for any set of n segments, there is a triangle to be a smallest-area convex T-covering of the set, and the triangle can be computed efficiently in $O(n \log n)$ time [1]. It is further conjectured that the smallest-area convex TR-covering of a family of triangles is a triangle, which is shown to be true for some families [23].

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The equilateral triangle of height 1 is the smallest-area convex T-covering of the set of all curves of unit length, as well as the unit line segments. In contrast to it, the problem of finding the smallest-area convex TR-covering of the set of all curves of unit length is notoriously difficult. The problem was given by Leo Moser as an open problem in 1966 [17], and it is still unsolved. The best lower bound of the smallest area is 0.21946 [27]. For the best upper bound, Wetzel informally conjectured (formally published in [28]) in 1970 that the 30° circular fan of unit radius, which has an area $\pi/12 \approx 0.2618$, is a convex TR-covering of all unit-length curves, and it was proved by Panraksa and Wichiramala [22]. Recently, the upper bound was improved to 0.260437 [20], but there still remains a substantial gap between the lower and upper bounds.

This problem is known as *Moser's worm problem*, and it has many variants. The history of progress on the topic can be found in an article [18] by William Moser (Leo's younger brother), in Chapter D18 in [8], and in Chapter 11.4 in [7]. It is interesting to find a new variant of Moser's worm problem with a clean mathematical solution.

Let us consider the set S_{c} of all closed curves of length 2. Here, we follow the tradition of previous works on this problem that deals with closed curves of length 2 instead of length 1, since a unit line segment can be considered as a degenerate convex closed curve of length 2. The problem to find a small-area convex covering of S_{c} is known to be an interesting but hard variant of Moser's worm problem, and it remains unsolved for T and TR despite of substantial efforts in the literature [10, 28, 25, 8, 7]. Wichiramala [29] showed that a hexagon obtained by clipping two corners of a rectangle is a convex TR-covering of S_{c}, which has area slightly less than 0.441. It is also shown that any convex TR-covering of S_{c} has area at least 0.39 [12], which has been recently improved to 0.4 [13] with help of computer programs. For convex T-coverings, the smallest area is known to be between 0.620 and 0.657 [7].

There are some works on restricted shapes of covering. Especially, if we consider triangular coverings, Wetzel [25, 26] gave a complete description, and it is shown that an acute triangle with side lengths a, b, c and area X becomes a T-covering (resp. TR-covering) of S_{c} if and only if $2 \leq \frac{8X^2}{abc}$ (resp. $2 \leq \frac{2\pi X}{a+b+c}$). As a consequence, the equilateral triangle of side length $4/3$ (resp. $\frac{2\sqrt{3}}{\pi}$) is the smallest triangular T-covering (resp. TR-covering) of S_{c}. Unfortunately, their areas are larger than those of the known smallest-area convex coverings.

Finite subgroups of the rotation group $R = SO(2, \mathbb{R})$ are cyclic groups $Z_k = \{e^{2i\pi\sqrt{-1}/k} \mid 0 \leq i \leq k-1\}$ for $k = 1, 2, \ldots$, where $e^{\theta\sqrt{-1}}$ means the rotation of angle θ. The group generated by T and Z_k is denoted by $G_k = T \rtimes Z_k$.

Recently, the convex coverings under the action of the group G_k was investigated by Jung *et al.* [14]. They showed that the smallest-area convex G_2-covering of S_{c} is the equilateral triangle of height 1, whose area is $\frac{\sqrt{3}}{3} \approx 0.577$. They also showed that the equilateral triangle with height $\beta = \cos(\pi/12) \approx 0.966$ is a convex G_4-covering of S_{c}. Its area is $\frac{2\sqrt{3}+3}{12} \approx 0.538675$, and it is conjectured to be the smallest-area convex G_4-covering. If the above conjecture is true, it is a curious phenomenon that the discrete rotations in G_2 and G_4 make the shape of the smallest-area convex covering of S_{c} simple and symmetric compared to the currently known small-area convex T-coverings and TR-coverings.

Among the convex G_3-coverings of S_{c} known so far, the smallest one has area 0.568 [14], and its shape is not a triangle.

There is another type of discrete groups of linear transformations from Z_k for which the shapes of geometric objects are invariant. They are dihedral groups D_k generated by the discrete rotation group Z_k and the reflection with respect to the x-axis. They have order $2k$. As groups, $D_1 \simeq Z_2$, $D_2 \simeq Z_2 \times Z_2$, and $D_3 \simeq S_3$, which is the symmetric group of degree 3. The dihedral group D_3 is also called the A_1-Weyl group as a reflection group. Therefore, it is natural to consider the group $H_k = T \rtimes D_k$ generated by the translation group T and the dihedral group D_k, which we call an *affine dihedral group*. Note that H_k is a subgroup of TO, but not a subgroup of TR.

Our results are as follows.

1. The smallest-area convex H_k-covering of the set S_{seg} of all unit segments is determined for each k.
2. The equilateral triangle of height 1 is an H_1-covering of the set S_{c} of all closed curves of length 2 if and only if it is located so that one of the edges is parallel to the x-axis.
3. The equilateral triangle given above is a minimal convex H_1-covering of S_{c}. It is a smallest-area triangle H_1-covering of S_{c}.
4. The equilateral triangle of height $\beta = \cos(\pi/12)$ is a minimal convex H_2-covering of S_{c} if it is located such that one of its sides has orientation $\pi/4$.
5. The trapezoid obtained by clipping the top corner of the equilateral triangle of height 1 with base parallel to the x-axis is a convex H_3-covering of S_{c}. The area of the covering is strictly smaller than that of the equilateral triangle of height $\beta = \cos(\pi/12)$.

We use elaborate but quite elementary geometric methods to show the results.

Here we introduce the notation and preliminaries. The orientation of a line is the angle swept from the

x-axis in a counterclockwise direction to the line, and it is thus in $[0, \pi)$. The orientation of a segment is the orientation of the line containing the segment. For two points X and Y, we use ℓ_{XY} to denote the line through X and Y. For a compact set U in the plane, we use $|U|$ to denote the area of U. If U is a line segment, then $|U|$ denotes the length of U.

The missing proofs of lemmas and corollaries can be found in the full version.

2 Universal convex coverings of line segments

Since we only consider convex coverings, we say covering for a convex covering from now on unless specifically noticed. We recall a result by Ahn *et al.* [1].

Theorem 1 ([1]) *For any set S of line segments, there exists a triangle that is a smallest-area convex T-covering of S. If S is a finite set and $|S| = n$, such a triangle can be computed in $O(n \log n)$ time.*

The G-orbit of a segment s in S is the set of segments in S to which s can be moved by the elements of a group G. We derive the following corollary from Theorem 1:

Corollary 2 *For any set S of line segments, there is a triangle that is the smallest-area convex H_k-covering.*

The orbit structure of the D_k-action on S_{seg} is not uniform; for example, each of the horizontal and vertical line segments is invariant under the D_2-action and forms a single-element orbit while other orbits have two segments. This contrasts to the Z_k-action, for which the orbit structure is uniform. Consequently, any rotated copy of a G_k-covering of S_{seg} is also a G_k-covering of S_{seg}, but there are cases where a rotated copy of an H_k-covering of S_{seg} is not an H_k-covering of S_{seg}.

Theorem 3 *If k is odd, the smallest area of H_k-coverings of S_{seg} is $\frac{1}{2}\sin\frac{\pi}{2k}$, and it is attained by any triangle $\triangle XYZ$ with horizontal bottom side XY of length 1 and height $\sin\frac{\pi}{2k}$ such that $\frac{\pi}{2} \le \angle X \le \frac{(2k-1)\pi}{2k}$. If k is even, the smallest area of H_k-coverings of S_{seg} is $\frac{1}{2}\sin\frac{\pi}{k}$, and it is attained by any triangle $\triangle XYZ$ with horizontal bottom side XY of length 1 and height $\sin\frac{\pi}{k}$ such that $\frac{\pi}{2} \le \angle X \le \frac{(k-1)\pi}{k}$.*

Proof. The set of orientations of segments in S_{seg} corresponds to the angle interval $[0, \pi)$. First, consider the case where k is odd. Each orbit of Z_k action has exactly k elements. Each of the horizontal and vertical segments has a single orbit in the orbit structure of the action of the reflection with respect to the x-axis while the other segments has 2 orbits in the same orbit structure. The orbit structure of D_k is given by the above combinations, and thus each orbit has at most $2k$ elements.

Let P be an H_k-covering of S_{seg}. For any segment s in S_{seg}, at least one segment in the D_k-orbit of s must be contained in P by translation. Let Y be the set of unit segments that can be contained in P by translation. Consider the smallest angle interval I such that for each segment of orientation θ in Y, $\theta \in I$ or $\theta + \pi \in I$. Observe that I has length at least $\frac{\pi}{2k}$. If the length of I is smaller than $\frac{\pi}{2k}$, the set of orientations of the segments in Y under H_k-action is a proper subset of $[0, \pi)$ since the D_k-orbit of a segment s' of Y has at most $2k$ elements. This contradicts that P is an H_k-covering of S_{seg}. So there are two segments in Y such that their intersection angle $\bar{\theta}$ (the one not larger than $\frac{\pi}{2}$) is not smaller than $\frac{\pi}{2k}$. The convex hull P' of the two segments has area $|P'| = \frac{1}{2}\sin\bar{\theta} \ge \frac{1}{2}\sin\frac{\pi}{2k}$, and $|P| \ge |P'|$. Thus, the smallest area of H_k-coverings is at least $\frac{1}{2}\sin\frac{\pi}{2k}$.

If k is even, each orbit of Z_k action has exactly $\frac{k}{2}$ elements. Thus, each orbit of D_k has at most k elements. The rest is analogous to the odd k case, and the smallest area of H_k-coverings is at least $\frac{1}{2}\sin\frac{\pi}{k}$.

Observe that the triangles given in the theorem are coverings with areas $\frac{1}{2}\sin\frac{\pi}{2k}$ for odd k and $\frac{1}{2}\sin\frac{\pi}{k}$ for even k. □

The triangles obtained by acting elements $h \in H_k$ and $g \in G_2$ on the triangles given in Theorem 3 are also H_k-coverings, since a line segment is invariant with respect to the action of G_2 and the covering condition is invariant with respect to the action of H_k.

Let us compare Theorem 3 with the G_k-coverings of S_{seg} given in [14]. If $k \ge 2$, the smallest area of H_k-coverings is the same as that of the smallest area of G_{2k}-coverings. The smallest area of H_1-coverings is $\frac{1}{2}$, which is the same as the smallest area of G_4-coverings. In contrast, the smallest-area G_2-covering (that is the same as the smallest-area T-covering) of S_{seg} is the equilateral triangle of area $\frac{1}{\sqrt{3}}$.

3 Universal convex H_1-coverings of S_{c}

In this section, we consider H_1-coverings of the set S_{c} of all closed curves of length 2. First, we recall known results mentioned in the introduction.

3.1 The smallest-area covering and related results

Theorem 4 (Pal's theorem) *The equilateral triangle of height 1 is the smallest-area (convex) T-covering of the set of all unit line segments.*

Corollary 5 *The area of a G_2-covering of S_{c} is at least $1/\sqrt{3}$.*

Proof. Observe that all unit line segments are in S_{c}, and line segments are stable under the action of rotation by π. Thus, any convex G_2-covering of S_c must be a T-covering of all unit line segments, and the corollary follows from Theorem 4 (Pal's theorem). $\square$

It is known that the above lower bound is tight.

Theorem 6 (Jung *et al.* [14]) *The equilateral triangle of height 1 is the smallest-area G_2-covering of S_c.*

Lemma 7 *Suppose that a region P is symmetric with respect to the y-axis. Then the following holds.*

- *For an odd k, P is an H_k-covering of S_c if and only if it is a G_{2k}-covering of S_c.*
- *For an even k, P is an H_k-covering of S_c if and only if it is a G_k-covering of S_c.*

Proof. Let h be the reflection with respect to the x-axis, and let g be the rotation of angle π about the origin. From the assumption that P is symmetric with respect to the y-axis, $h \cdot P$ is a translation of $g \cdot P$. If k is even, G_k contains g. Hence the H_k-orbit of P is the same as the G_k-orbit of P, and we have the second statement. If k is odd, the group generated by G_k and g is G_{2k}, and we have the first statement. $\square$

3.2 H_1-coverings of S_c

Let $\triangle_1$ be an equilateral triangle of height 1 whose bottom side is horizontal.

Theorem 8 *The equilateral triangle $\triangle_1$ is an H_1-covering of S_c. Moreover, it is the smallest-area H_1-covering among all H_1-coverings of S_c that are convex and symmetric to the y-axis.*

Proof. The first statement follows immediately from Theorem 6 and Lemma 7. Consider a H_1-covering P that is convex and symmetric with respect to the y-axis. By Lemma 7, P is G_2-covering. By Theorem 6, $|\triangle_1| \leq |P|$. Thus, the second statement also holds. $\square$

Corollary 9 *Any closed curve of length 2 that is symmetric with respect to a line of orientation 0, $\pi/3$ or $2\pi/3$ is contained in $\triangle_1$ by translation.*

This corollary complements the fact that any centrally-symmetric closed curve of length 2 can be contained in $\triangle_1$ by translation [14]. However, $\triangle_1$ is not the smallest-area T-covering of the set of the closed curves of length 2 that are symmetric about the x-axis, since a square with a unit length horizontal diagonal is a T-covering of the set. The area of the square is $\frac{1}{2}$, and thus smaller than that of $\triangle_1$. Thus, the H_1-covering problem and the T-covering problem of D_1-invariant objects have different solutions.

Theorem 10 *Let T_L be an equilateral triangle of perimeter 2 such that it has a vertical side and its opposite corner lies to the right. Let T_R be a copy of T_L rotated by π. The equilateral triangle $\triangle_1$ and its reflected image about the x-axis are the only convex H_1-coverings of T_L and T_R among the rotated copies of $\triangle_1$ about the origin.*

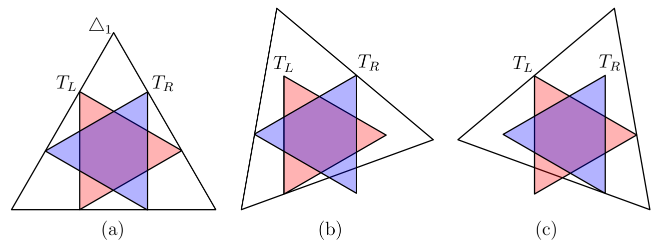

Figure 1: (a) Translates of T_L and T_R that are contained in $\triangle_1$. (b) No translate of T_R can be contained in a rotated copy of $\triangle_1$ by θ with $0 < \theta < \pi/3$. (c) No translate of T_L can be contained in a rotated copy of $\triangle_1$ by θ with $-\pi/3 < \theta < 0$.

Proof. Observe that there are translates of T_L and T_R that are contained in $\triangle_1$. See Figure 1 (a). Observe that both T_L and T_R are symmetric with respect to a horizontal line. If a rotated copy $\triangle_\theta$ of $\triangle_1$ by θ is a H_1-covering of T_L and T_R, there are translates of T_L and T_R that are contained in $\triangle_\theta$. Observe that no translate of T_R is contained in $\triangle_\theta$ with $0 < \theta < \pi/3$ and no translate of T_L is contained in $\triangle_\theta$ with $-\pi/3 < \theta < 0$, as shown in Figure 1 (b) and (c). $\triangle_1$ is invariant under rotation by $2\pi/3$. The rotation by $\pi/3$ of $\triangle_1$ is equivalent to $g \cdot \triangle_1$, where g is the reflection with respect to the x-axis. Thus, $\triangle_1$ and $g \cdot \triangle_1$ are the only convex H_1-coverings of T_L and T_R among the rotated copies of $\triangle_1$. $\square$

3.3 The minimality of the H_1-covering $\triangle_1$

One may wonder whether we may remove some part of $\triangle_1$ to obtain a smaller H_1-covering of S_c. In this section, we prove the minimality of the H_1-covering $\triangle_1$.

Theorem 11 *The equilateral triangle $\triangle_1$ is a minimal convex H_1-covering of S_c.*

Proof. Assume to the contrary that there is a proper subset T of $\triangle_1$ that is a convex H_1-covering of S_c. Since $\triangle_1$ is the convex hull of the corners of $\triangle_1$, T must be obtained by clipping some portions around some corners of $\triangle_1$. Since a vertical unit segment must be contained in T, no portion around the top corner of $\triangle_1$ can be cut off.

Let I_n be an isosceles triangle with perimeter 2 whose legs are of $1 - 1/3n$ each, base is parallel to x-axis, and apex is the top corner of I_n. Let I'_n be a copy of

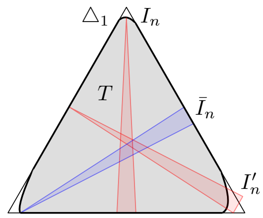

Figure 2: No proper subset T of $\triangle_1$ is a convex H_1-covering of S_c. The isosceles triangle I_n, a rotated copy I'_n of I_n by $\pi/3$, and a reflected copy $\bar{I}_n$ of I'_n along the x-axis.

I_n rotated by $\pi/3$. Observe that no translate of I'_n is contained in $\triangle_1$ for any positive integer n. See Figure 2. Since T is a proper subset of $\triangle_1$, no translate of I'_n is contained in T for any positive integer n. Thus, a reflected copy of I'_n along the x-axis is contained in T under translation for every n.

Let $\bar{I}_n$ be the reflection copy of I'_n such that $\bar{I}_n$ is contained in T. Since T is compact, there is a subsequence $\{\bar{I}_{n_i}\}$ that converges to a unit line segment with orientation $\pi/6$ contained in T. Observe that $\triangle_1$ contains a unit line segment with orientation $\pi/6$ only when the left endpoint of the segment lies at the bottom-left corner of $\triangle_1$. Thus, no portion around the bottom-left corner of $\triangle_1$ can be cut off. Similarly, no portion around the bottom-right corner of $\triangle_1$ can be cut off. Therefore, we conclude that that T is $\triangle_1$, contradicting that T is a proper subset of $\triangle_1$. Thus, $\triangle_1$ is a minimal H_1-covering of S_c. □

3.4 The smallest area triangle H_1-covering of S_c

Now we show that $\triangle_1$ has the smallest area among all triangle H_1-coverings of S_c. The following two lemmas describe geometric properties of a triangle that circumscribes a convex polygon P.

Lemma 12 ([16]) *If a triangle T has a local minimum in area among all triangles enclosing a convex polygon P, the midpoint of each side of T touches P.*

Following [19], we say that a side s of a triangle is *flush* with an edge e of P if $e \subseteq s$. Also, we say that a circumscribing triangle $\triangle$ is *P-anchored* if a side of $\triangle$ is flush with an edge of P and the other two sides of $\triangle$ touch vertices of P at their midpoints.

Lemma 13 (Lemma 1 of [19]) *For any P-anchored triangle $\triangle$, there always exists some P-anchored triangle $\triangle'$ such that $|\triangle| = |\triangle'|$, $\triangle$ and $\triangle'$ share one side, and at least two sides of $\triangle'$ are flush with edges of P.*

Recall that T_L given in Theorem 10 is an equilateral triangle of perimeter 2 such that it has a vertical side and its opposite corner lies to the right and T_R is a copy of T_L rotated by π.

Lemma 14 *Let Q be the convex hull of T_L and a translated copy of T_R. Then $|Q| \geq |T_L| + |T_R|$.*

The following lemma can be shown by Lemmas 12, 13, and 14.

Lemma 15 *The equilateral triangle $\triangle_1$ is the smallest triangle T-covering of T_L and T_R.*

Theorem 16 *The equilateral triangle $\triangle_1$ is the smallest-area triangle H_1-covering of S_c.*

Proof. Let $\triangle$ be a smallest-area triangle H_1-covering of S_c. Since $\triangle$ is H_1-covering, it is a covering of T_L and T_R under translation. By Lemma 15, $\triangle_1$ is the smallest-area triangle H_1-covering of S_c. □

A major open problem is whether $\triangle_1$ is a smallest-area H_1-covering of S_c. As Theorem 6 says, $\triangle_1$ is the smallest-area G_2-covering of S_c, and it is because $\triangle_1$ is the smallest-area G_2-covering of S_seg. However, as we have seen in Theorem 3, the smallest-area H_1-covering of S_seg is smaller, and has area 1/2. This is because a line segment (located so that its midpoint is at the origin) is stable under the rotation by π, but not stable under the reflection with respect to the x-axis unless it is horizontal or vertical.

4 Universal convex H_2-coverings of S_c

The dihedral group D_2 is generated by the reflection ρ with respect to the x-axis and the π-rotation g. Note that $g\rho$ is the reflection with respect to the y-axis.

The following result was given by Jung *et al.* [14]:

Theorem 17 *An equilateral triangle of height $\beta = \cos(\pi/12) \approx 0.966$ is a G_4-covering of S_c.*

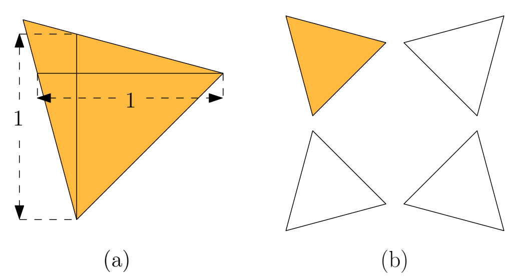

Figure 3: (a) An equilateral triangle $\triangle_\beta$ of height β containing horizontal and vertical unit line segments. (b) The triangle $\triangle_\beta$ and three triangles forming the D_2-orbit of $\triangle_\beta$.

Now, we consider $\triangle_\beta$ that is the equilateral triangle of height β located such that one of its sides has orientation $\pi/4$. Then, we have the following:

Theorem 18 *The equilateral triangle $\triangle_\beta$ is an H_2-covering of S_{c}. Moreover, it is minimal.*

Proof. Consider the D_2-orbit of $\triangle_\beta$. Then, as observed in Figure 3, they are exactly the same as the rotated copies of $\triangle_\beta$ with $k\pi/2$-rotations for $k = 0, 1, 2, 3$. Thus, it follows from Theorem 17 that $\triangle_\beta$ is an H_2-covering.

Consider the set A of unit length segments (regarded as degenerate closed curves of length 2) that are contained in $\triangle_\beta$. It is observed that $A' = A \setminus B$ has at most one element of each D_2-orbit of S_{seg}, where B is the set of six segments with orientations $k\pi/6$ for $k = 0, 1, \ldots, 5$. Thus, each segment in A' must be contained in any H_2-covering $Q \subseteq \triangle_\beta$ under translation. By Theorem 1, there is a smallest-area T-covering of A' that is a triangle, and the algorithm given in [1] shows that $\triangle_\beta$ is the triangle. Thus, $Q = \triangle_\beta$, and hence $\triangle_\beta$ is minimal. □

We say that an object is *θ-orthogonal symmetric* if it has a pair of symmetry axes with orientations θ and $\theta + \pi/2$.

Corollary 19 *Any curve in S_{c} that is θ-orthogonal symmetric for either $\theta = 0, \pi/3$, or $2\pi/3$ can be contained in $\triangle_\beta$ by translation. In particular, any rectangle of perimeter 2 that has an edge with orientation either 0, $\pi/3$, or $2\pi/3$ can be contained in $\triangle_\beta$ by translation.*

Note that $\triangle_1$ is the smallest-area T-covering of the family of all rotated rectangles of perimeter 2 [14].

5 Universal convex H_3-coverings of S_{c}

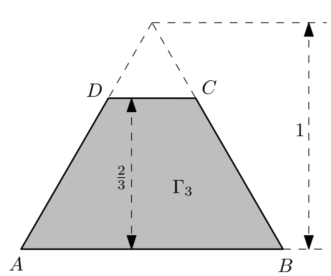

Figure 4: Construction of a convex H_3-covering Γ_3 of S_{c}. It is the trapezoid obtained from $\triangle_1$ after clipping a top part (an equilateral triangle) of height 1/3.

Let Γ_3 be the trapezoid obtained from $\triangle_1$ after clipping a top part (an equilateral triangle) of height 1/3. See Figure 4. Let A, B, C, D be the corners of Γ_3 in counterclockwise order, with A being the bottom-left corner. We show that Γ_3 is a convex H_3-covering of S_{c}.

A *slab* is the region bounded by two parallel lines in the plane, and its width is the distance between the lines. Let L_θ denote a slab of orientation θ, and let $w(L_\theta)$ be the width of L_θ.

Lemma 20 *For a closed curve β, let L_θ be the minimum-width slab of orientation θ that contains β. The length of β is at least $w(L_0) + w(L_{\pi/3}) + w(L_{2\pi/3})$.*

By Lemma 20, we have the following result.

Theorem 21 *The trapezoid Γ_3 is a convex H_3-covering of S_{c}.*

The area of Γ_3 is strictly smaller than that of the equilateral triangle of height $\beta = \cos(\pi/12)$.

6 Conclusion

This research is about how the mirror (i.e., reflection) effects on Moser's worm problems. Compared to the discrete rotation case given in [14], the positioning of the covering matters if we introduce the reflection, which requires delicate mathematical handling.

The research status of Moser's worm problems on S_{c} for T and TR remains rather static, and it is awkward to conjecture that the known small-area coverings are the optimal ones. In contrast to it, if we consider the affine dihedral groups such as H_1 and H_2, we can give clear conjectures that suitable equilateral triangles are the smallest-area coverings. The authors think they are mathematically clean and curious conjectures, and hope novel mathematical tools will be developed in the course of challenging to prove or disprove them.

Finally, although the H_k-coverings for $k \geq 4$ of S_{seg} have been classified, those for S_{c} have not been investigated, and it would be curious to find a unified approach to study them.

References

[1] H.-K. Ahn, S.-W. Bae, O. Cheong, J. Gudmundsson, T. Tokuyama, A. Vigneron, A Generalization of the Convex Kakeya Problem, *Algorithmica,* **70**:152-170, 2014.

[2] H.-K. Ahn, O. Cheong, Aligning Two Convex Figures to Minimize Area or Perimeter, *Algorithmica,* **62**:464-479, 2012.

[3] S. W. Bae, S. Cabello, O. Cheong, Y. Choi, F. Stehn, S. D. Yoon: The reverse Kakeya problem, *Advances in Geometry*, **21(1)**:75-84, 2021.

[4] J. Baez, K. Bagdasaryan, P. Gibbs, The Lebesgue universal covering problem, *Journal of Computational Geometry,* **6(1)**:288-299, 2015.

[5] A. Besicovitch, On Kakeya's problem and a similar one, *Mathematische Zeitschrift,* **27**:312-320, 1928.

[6] K. Bezdek, R. Connelly, Covering Curves by Translates of a Convex Set, *The American Mathematical Monthly*, **96(9)**:789-806, 1989.

[7] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer Verlag, 2005.

[8] H. T. Croft, K.J. Falconer, R.K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, 1991.

[9] Z. Dvir, On the size of Kakeya sets in finite fields. *J. Amer. Math. Soc.* **22**(4): 1093-1097, 2009.

[10] Z. Fürdi, J. W. Wetzel, Covers for Closed Curves of Length Two, *Periodia Mathematica Hungaria,* **63(1)**:1-17, 2011.

[11] P. Gibbs, An Upper Bound for Lebesgue's Covering Problem, arXiv:1810.10089, 2018.

[12] B. Grechuk, S. Som-am, A convex cover for closed unit curves has area at least 0.0975, *International Journal of Computational Geometry & Applications,* **30(2)**:121-139, 2020.

[13] B. Grechuk, S. Som-am, A convex cover for closed unit curves has area at least 0.1, *Discrete Optimization,* **38**, article 100608: 1-15, 2020.

[14] M. K. Jung, S. D. Yoon, H.-K. Ahn, T. Tokuyama, Universal convex covering problems under translation and discrete rotations, arXiv:2211.14807 [cs.CG] , 2022.

[15] S. Kakeya, Some problems on maximum and minimum regarding ovals, *Tohoku Science Report,* **6**:71-88, 1917.

[16] V. Klee, Facet-centroids and volume minimization, *Studia Scientiarum Mathematicarum Hungarica*, **21**:143-147 1986.

[17] L. Moser, Poorly formulated unsolved problems of combinatorial geometry, Mimeographed, 1966.

[18] W. O. J. Moser, Problems, problems, problems, *Discrete Applied Mathematics,* **31**:201-225, 1991.

[19] J. O'Rourke, A. Aggarwal, S. Maddila, M. Baldwin, An Optimal Algorithm for Finding Minimal Enclosing Triangles, *Journal of Algorithms,* **7**:258-269, 1986.

[20] R. Norwood, G. Poole, An improved upper bound for Leo Moser's worm problem, *Discrete Computational Geometry,* **29**:409-417, 2003.

[21] J. Pal, Ein Minimumproblem für Ovale, *Mathematische Annalen,* **83**:311-319, 1921.

[22] C. Panraksa, W. Wichiramala, Wetzel's sector covers unit arcs, *Periodica Mathematica Hungarica,* **82**:213-222, 2021.

[23] J. Park, O. Cheong, Smallest universal covers for families of triangles. *Computational Geometry,* **92**: 101686, 2021.

[24] T. Tao, From Rotating Needles to Stability of Waves: Emerging Connections between Combinatorics, Analysis, and PDE, *Notice of American Mathematical Society*, **48**(3): 294-303, 2001.

[25] J. E. Wetzel, Triangular covers of closed curves of constant length, *Elemente der Mathematik,* **25(4)**:78-82, 1970.

[26] J. E. Wetzel, On Moser's problem of accommodating closed curves in triangle, *Elemente der Mathematik,* **27(2)**:35-36, 1972.

[27] J. E. Wetzel, Sectorial covers for curves of constant length, *Canadian Mathematical Bulletin,* **16**:367-375, 1973.

[28] J. E. Wetzel, Fits and covers. *Mathematics Magazine,* **76**:349-363, 2003.

[29] W. Wichiramala, A smaller cover for closed unit curves, *Miskolc Mathematical Notes,* **19(1)**:691-698, 2018.