Aligning two convex figures to minimize area or perimeter*

Hee-Kap Ahn[1] Otfried Cheong[2]

December 15, 2009

Abstract

Given two compact convex sets P and Q in the plane, we consider the problem of finding a placement φP of P that minimizes the convex hull of $\varphi P \cup Q$. We study eight versions of the problem: we consider minimizing either the area or the perimeter of the convex hull; we either allow φP and Q to intersect or we restrict their interiors to remain disjoint; and we either allow reorienting P or require its orientation to be fixed. In the case without reorientations, we achieve exact near-linear time algorithms for all versions of the problem. In the case with reorientations, we compute a $(1+\varepsilon)$-approximation in time $O(\varepsilon^{-1/2} \log n + \varepsilon^{-3/2} \log^a(1/\varepsilon))$ if the two sets are convex polygons with n vertices in total, where $a \in \{0, 1, 2\}$ depending on the version of the problem.

1 Introduction

We consider the problem of stacking two flat objects into a box. The objects will lie on top of each other, and our goal is to design a box that is as small as possible, while restricting it to be convex. This problem can be modelled as follows: given two planar compact convex figures P and Q, find the rigid motion φ such that the convex hull of $\varphi P \cup Q$ is minimized. Note that considering convex sets only is no restriction, as the target does not change when we replace a set by its convex hull.

Problems of this flavor arise in various applications, consider for instance the problem of minimizing the cross section of a wire bundle consisting of two subsets of wires. Depending on the application, we may want to minimize either the area or the perimeter of the convex hull. We may allow the two figures to overlap (as in the case where we design a box for two stacked objects), or not (as in the case of the wire bundle). Finally, we may allow the objects to be reoriented freely, or we may consider their orientation to be fixed. We study all these eight versions of the problem in this paper.

There has been a fair amount of work on the problem of *maximizing the overlap* (or, equivalently, minimizing the symmetric difference) of two shapes in the context of shape matching under translations [6, 9, 18] or rigid motions [4, 5, 8].

Much less is known about the problem of *minimizing the convex hull* of two shapes. The problem has been studied only under the restriction that the interiors of the two shapes remain disjoint. Lee and Woo [13] presented a linear time algorithm to find a translation that minimizes the area of the convex hull of two convex polygons with n vertices in total. Tang et al. [20] recently gave an $O(n^3)$ time deterministic algorithm to find a rigid motion minimizing this convex hull.

In higher dimensions, Ahn et al. [3] studied the problem of minimizing the volume of the convex hull of two convex polyhedra under translations when overlap is allowed. Their algorithm returns the optimal translation in expected time $O(n^{d+1-3/d} \log^{d+1} n)$ for dimension $d \geqslant 3$.

Motivated by applications in the clothing industry, Milenkovic studied the packing of a set of polygons into another polygon container under translations [17] and rigid motions [15, 16]. For packing k convex n-gons into a minimum area isothetic rectangle under translation, he gave an $O(n^{k-1} \log n)$ time algorithm using linear programming techniques. Later, Alt and Hurtado [7] considered the problem of packing convex polygons into a minimum size rectangle, where the *size* of a rectangle can either be its area or its perimeter. When overlap is allowed, they presented efficient algorithms whose running time is close

*H.-K. Ahn was supported by the National Research Foundation of Korea Grant funded by the Korean Government (MEST) (NRF-2009-0067195) . O. Cheong was supported by the Korea Science and Engineering Foundation Grant R01-2008-000-11607-0 funded by the Korean government.

[1]Dept. of Computer Science and Engineering, POSTECH, Pohang, Korea. heekap@postech.ac.kr

[2]Dept. of Computer Science, KAIST, Daejeon, Korea. otfried@kaist.edu

to linear or $O(n \log n)$ even for an arbitrary number of polygons with n vertices in total. For disjoint packing of two polygons, they presented algorithms whose running time is close to linear for translations and $O(n^3)$ for rigid motions. This problem is known to be NP-hard for arbitrary numbers of polygons. Very recently, Egeblad et al. [12] consider the problem of translationally packing polytopes into another polytope container in arbitrary dimension.

Sugihara et al. [19] considered a disk packing problem of finding the smallest enclosing circle containing a set of disks (the cross section of a wire bundle) in the plane, and gave a “shake-and-shrink” algorithm that shakes the disks and shrinks the enclosing circle step-by-step.

Ahn et al. [2] gave an approximation algorithm to find the line ℓ such that the convex hull of a given convex set C and a reflected copy of C along ℓ is minimized. This is a special case of one version of our problem, but does not generalize to the more general problem.

Our main theoretical result is to show that for any two given convex figures P and Q, the function $\omega : \mathbb{R}^2 \to \mathbb{R}$ that maps a vector r to either the area or the perimeter of the convex hull of $P + r$ and Q, is a convex function. For the perimeter case, we actually prove this result for arbitrarily large families of convex figures (for the area case, however, the statement is false for more than two figures).

This implies that minimizing either the area or the perimeter of two convex figures without reorientations can be solved numerically by minimizing a convex function. More theoretically, we show that when the two figures are convex polygons with n vertices in total, the area can be minimized in time $O(n \log n)$, while the perimeter can be minimized in time $O(n \log^2 n)$.

We then give an ε-approximation algorithm for the problem when reorientations are allowed. More precisely, we compute a rigid motion φ^{app} such that either the area or the perimeter of the convex hull of $\varphi^{\text{app}} P \cup Q$ is at most $1 + \varepsilon$ times the optimally achievable area or perimeter. The running time is $O(\varepsilon^{-1/2} T + \varepsilon^{-3/2} \log^a(1/\varepsilon))$, where T is the time needed to find a point extreme in a given direction, or the intersection of a given line with P and Q, and where $a = 1$ for the area case and $a = 2$ for the perimeter case. Our approximation algorithm is simple: it suffices to sample orientations of P, and to find the optimal translation for each orientation of P using the algorithm developed earlier. The difficulty is in choosing a suitable set of sampling orientations for P—uniform sampling does not work.

We also briefly revisit the case where P and Q are not allowed to overlap, and give similar approximation algorithms to find the best rigid motion minimizing area or perimeter of the convex hull under this restriction.

2 Notation and preliminaries

For a compact set S in the plane, let $d(S)$ and $w(S)$ denote the diameter and width of S (the width is the minimum distance between two parallel lines enclosing S). Also, let $|S|$ and $||S||$ denote the perimeter and the area of *the convex hull of* S. Note that $d(S) = d(\text{conv}(S))$, $w(S) = w(\text{conv}(S))$, and (by definition) $|S| = |\text{conv}(S)|$ and $||S|| = ||\text{conv}(S)||$. For a vector $r \in \mathbb{R}^2$, let $S + r$ denote the translation of S by r, that is, $S + r = \{p + r \mid p \in S\}$. The interior of a set $S \subseteq \mathbb{R}^2$ is denoted $\text{int}(S)$.

For an angle θ, let $w(S, \theta)$ be the width of S in direction θ, that is, the length of the orthogonal projection of S onto a line with slope θ. Clearly we have $w(S) = \min_{0 \leqslant \theta < \pi} w(S, \theta)$. We will make use of (a special case of) the Cauchy-Crofton formula [10], which asserts that

$$|S| = \int_0^\pi w(S, \theta) d\theta. \tag{1}$$

Let $C \subset \mathbb{R}^2$ be a compact convex set. We will need the inequality $|C| \leqslant \pi d(C)$ [21, pg. 257, ex.7.17a]. If C' is the set of points at distance at most $r > 0$ from C (in other words, C' is the Minkowski sum of C and a disk of radius r), then [2]

$$|C'| = |C| + 2\pi r \tag{2}$$

$$||C'|| = ||C|| + r|C| + \pi r^2. \tag{3}$$

Following Agarwal et al. [1], we say that a convex set $C' \subset C$ is an ε-*kernel* of C if and only if for all $\theta \in [0, \pi]$ we have $(1 - \varepsilon) w(C, \theta) \leqslant w(C', \theta)$. By Eq. (1), this immediately implies that

$$(1 - \varepsilon)|C| \leqslant |C'| \leqslant |C|. \tag{4}$$

There is a constant $c_2 > 1$ such that if C' is an ε/c_2-kernel of C, then $||C \setminus C'|| \leqslant \varepsilon||C||$ [1], which additionally implies

$$(1-\varepsilon)||C|| \leqslant ||C'|| \leqslant ||C||. \tag{5}$$

Based on Dudley's constructive proof from 1974 [11], Ahn et al. [2] gave an algorithm that computes inner and outer approximations to a given convex set C. The algorithm requires only two operations on the set C, namely (a) given a query direction $u \in \mathcal{U}$, find an extreme point in direction u; and (b) given a line ℓ, find the line segment $\ell \cap C$. Let T_C denote the time needed to answer any of these queries.

Lemma 1. *([2]) Given a planar convex set C and $\varepsilon > 0$, one can construct in time $O(T_C/\sqrt{\varepsilon})$ a convex polygon C' with $O(1/\sqrt{\varepsilon})$ vertices such that $C \subset C'$ and C is an ε-kernel of C'. If C is a convex n-gon whose vertices are given in a sorted array, then we can compute C' in time $O(\log n/\sqrt{\varepsilon})$.*

3 Convexity of the target functions

Let $k \geqslant 2$ be an integer, and let C_i, for $i = 1, \dots, k$, be k compact convex figures in the plane. For a vector $r \in \mathbb{R}^{2k}$, we write $r = (r_1, \dots, r_k)$, where $r_i \in \mathbb{R}^2$, let $C_i(r) := C_i + r_i$, and let $C(r) := \operatorname{conv}\bigl(\bigcup_i C_i(r)\bigr)$. We can now define two functions ψ, $\omega : \mathbb{R}^{2k} \to \mathbb{R}$, where $\psi(r) := |C(r)|$ and $\omega(r) := ||C(r)||$.

In this section, we show that ψ is a convex function, that is, the volume above the graph of the function is convex. We also show that ω is a convex function for $k = 2$, but that this is not necessarily the case for $k > 2$.

3.1 Convexity of the perimeter function

Theorem 2. *The function $\psi : \mathbb{R}^{2k} \to \mathbb{R}$ with $\psi(r) = |C(r)|$ is convex.*

Proof. Since a function $\mathbb{R}^{2k} \to \mathbb{R}$ is convex if any cross section along a line is convex, it suffices to prove the latter fact. We can express any line in $\mathbb{R}^{2k}$ as $r(t) = (1-t)u + tv$, where $u, v \in \mathbb{R}^{2k}$. We let $C_i(t) := C_i(r(t))$ and $C(t) := C(r(t)))$. We will show that the function $t \mapsto \psi(r(t)) = |C(t)|$ is convex, proving the theorem.

By the Cauchy-Crofton formula, we have

$$|C(t)| = \int_0^\pi w(C(t), \theta) d\theta.$$

We will show that for any fixed $\theta \in [0, \pi]$, the function $t \mapsto w(C(t), \theta)$ is convex. Without loss of generality, we can consider $\theta = 0$, so that $w(C(t), \theta)$ is the length of the projection of $C(t)$ onto the x-axis. Let a_i and b_i be a leftmost and rightmost point of C_i (that is, a point with the smallest x-coordinate and a point with the largest x-coordinate). Then $a_i + r_i(t)$ is a leftmost point of $C_i(t)$, and $b_i + r_i(t)$ is a rightmost point of $C_i(t)$. Let $\alpha_i(t)$ and $\beta_i(t)$ denote the x-coordinates of $a_i + r_i(t)$ and $b_i + r_i(t)$. Since $r(t)$ is a linear function, so are $\alpha_i(t)$ and $\beta_i(t)$. We now observe that

$$w(C(t), \theta) = \max_i \beta_i(t) - \min_i \alpha_i(t).$$

The pointwise maximum of a family of linear functions is a convex function, and so $t \mapsto \max_i \beta_i(t)$ and $t \mapsto -\min_i \alpha_i(t)$ are both convex. This implies that $t \mapsto w(C(t), \theta)$ is indeed a convex function.

Convexity of $t \mapsto w(C(t), \theta)$ means that for any $0 \leqslant \theta < \pi$, any $t_0, t_1 \in \mathbb{R}$, and any $0 \leqslant \lambda \leqslant 1$ we have

$$w\bigl(C\bigl((1-\lambda)t_0 + \lambda t_1\bigr), \theta\bigr) \leqslant (1-\lambda) \cdot w(C(t_0), \theta) + \lambda \cdot w(C(t_1), \theta).$$

Integrating over θ, we get

$$\begin{aligned} \bigl|C((1-\lambda)t_0 + \lambda t_1)\bigr| &= \int_0^\pi w\bigl(C((1-\lambda)t_0 + \lambda t_1), \theta\bigr) d\theta \\ &\leqslant (1-\lambda) \int_0^\pi w(C(t_0), \theta) d\theta + \lambda \int_0^\pi w(C(t_1), \theta) d\theta \\ &= (1-\lambda)|C(t_0)| + \lambda |C(t_1)|, \end{aligned}$$

which shows that $t \mapsto |C(t)|$ is convex and the theorem follows. □

3.2 Convexity of the area function

We now consider the function $\omega : \mathbb{R}^{2k} \to \mathbb{R}$ where $\omega(r) = ||C(r)||$ is the area of $C(r)$. This function is not necessarily convex for $k \geqslant 3$, as the following example shows.

Let $C_1 = C_2 = C_3$ be a disk of diameter $\varepsilon > 0$ centered at the origin, and consider the two vectors $u = (0, 0, 0, 2, 0, 6)$ and $v = (6, 0, 2, 0, 0, 0)$. Then $||C(u)||$ and $||C(v)||$ are both roughly 6ε. However, for the vector $w = \frac{1}{2}(u + v)$ halfway between u and v, we have $w = (3, 0, 1, 1, 0, 3)$, and $||C(w)||$ is roughly 3/2. If ε is chosen small enough, this violates the convexity of $r \mapsto ||C(r)||$.

Perhaps surprisingly, for $k = 2$ the area function *is* a convex function.

Theorem 3. *Let P and Q be compact convex figures in the plane. Then the function $\omega : \mathbb{R}^2 \to \mathbb{R}$ with $r \mapsto ||(P + r) \cup Q||$ is convex. If P and Q are convex polygons, then ω is piecewise linear.*

Proof. As in Theorem 2, it suffices to prove convexity along an arbitrary line. Without loss of generality, we consider horizontal lines. For $t \in \mathbb{R}$, let $P_t := P + (t, 0)$, that is, the translation of P by t along the x-axis, and let $C_t := \operatorname{conv}(P_t \cup Q)$. We will show that the function $\omega : t \mapsto ||C_t||$ is convex.

We first argue that it is enough to prove the lemma under the assumption that P and Q are convex *polygons*. Indeed, assume that the lemma is false for two compact convex figures P and Q. Then there are $t_0, t_1 \in \mathbb{R}$ and $0 < \lambda < 1$ such that $\omega\big(\lambda t_0 + (1-\lambda)t_1\big) > \lambda \cdot \omega(t_0) + (1-\lambda) \cdot \omega(t_1)$. Let $t^* := \lambda t_0 + (1-\lambda)t_1$, let $\varepsilon := \omega(t^*) - \big(\lambda \cdot \omega(t_0) + (1-\lambda) \cdot \omega(t_1)\big) > 0$, and let $\varepsilon' := \frac{\varepsilon}{2c_2\omega(t^*)}$, where $c_2 > 1$ is the constant needed for Ineq. (5).

We pick convex polygons $P' \subset P$ and $Q' \subset Q$ to be ε'-kernels of P and Q, and define $P'_t := P' + (t, 0)$. Then $\operatorname{conv}(P'_{t^*} \cup Q')$ is an ε'-kernel of C_{t^*}, and so $(1 - \frac{\varepsilon}{2\omega(t^*)})\omega(t^*) \leqslant ||P'_{t^*} \cup Q'||$ by (5). This implies $||P'_{t^*} \cup Q'|| \geqslant \omega(t^*) - \varepsilon/2 > \lambda \cdot \omega(t_0) + (1 - \lambda) \cdot \omega(t_1)$. On the other hand, $P'_t \cup Q' \subset C_t$, and so $||P'_t \cup Q'|| \leqslant \omega(t)$ for $t = t_0, t_1$. It follows that the function $t \mapsto ||P'_t \cup Q'||$ is not convex either, and so in the following we can assume that P and Q are convex polygons.

The vertices of C_t are either vertices of P_t or vertices of Q. Consequently, the boundary of C_t consists of three types of edges: edges of P_t (type 0 edges), edges of Q (type 2 edges), and type 1 edges connecting one vertex of P_t with one vertex of Q. As t increases, the polygon P_t moves to the right, while Q remains stationary. We consider how $||C_t||$ changes. As long as the combinatorial structure of C_t remains the same, the change in $||C_t||$ can be expressed as the sum of changes incurred by the individual edges: moving edges on the right side of C_t add area to C_t, while moving edges on the left side of C_t remove area from C_t. For a type 0 edge e, the area swept over when t increases by δ is $\delta h(e)$, where $h(S) = w(S, \pi/2)$ denotes the length of the projection of a set S on the y-axis. For a type 1 edge e, the area swept over is $\delta h(e)/2$ (since one endpoint moves by δ, the area swept is a triangle). Type 2 edges are stationary, and do not contribute to the change in $||C_t||$.

It follows that as long as $C_{t+\delta}$ and C_t have the same combinatorial structure, we have

$$\omega(t + \delta) - \omega(t) = ||C_{t+\delta}|| - ||C_t|| = \sum_{e \in R_t^0} \delta h(e) + \sum_{e \in R_t^1} \frac{\delta h(e)}{2} - \sum_{e \in L_t^0} \delta h(e) - \sum_{e \in L_t^1} \frac{\delta h(e)}{2},$$

where R_t^i is the set of right edges of C_t of type i, and L_t^i is the set of left edges of C_t of type i. Now note that $\sum h(e) = h(C)$, when the sum is taken over all left edges or over all right edges. It follows that

$$\sum_{e \in R_t^1} h(e) = h(C) - \sum_{e \in R_t^0} h(e) - \sum_{e \in R_t^2} h(e),$$
$$\sum_{e \in L_t^1} h(e) = h(C) - \sum_{e \in L_t^0} h(e) - \sum_{e \in L_t^2} h(e).$$

Substituting into the previous equality we obtain

$$\omega(t + \delta) - \omega(t) = \frac{\delta}{2}\bigg(\sum_{e \in R_t^0} h(e) - \sum_{e \in R_t^2} h(e) - \sum_{e \in L_t^0} h(e) + \sum_{e \in L_t^2} h(e) \bigg).$$

The expression in parentheses is constant as long as the combinatorial structure of C_t does not change. This implies that ω is a piecewise linear function.

Letting $\delta \to 0$, we obtain

$$\omega'(t) = \frac{1}{2}\Big(\sum_{e \in R_t^0} h(e) - \sum_{e \in R_t^2} h(e) - \sum_{e \in L_t^0} h(e) + \sum_{e \in L_t^2} h(e) \Big).$$

Consider a left edge e of P_t. This edge appears as a type 0 edge of C_t if and only if no vertex of Q lies on the left of the supporting line ℓ_e of e. Since P_t moves horizontally rightwards, there is a unique *edge event* for e where the moving ℓ_e *touches* the first vertex v_e of the stationary Q. Let $t(e)$ be the "time" of this event, that is, the value of t such that $v_e \in \ell_e$. Clearly, $e \in L_t^0$ if $t < t(e)$, and $e \notin L_t^0$ if $t > t(e)$ (and either can be true at $t = t(e)$). See Fig. 1. This implies that the function $t \mapsto \sum_{e \in L_t^0} h(e)$ is non-increasing. Similarly, a right edge e of P_t is in R_t^0 if and only if no vertex of Q lies to the right of ℓ_e. This implies that there is a "time" $t(e)$ such that $e \notin R_t^0$ if $t < t(e)$ and $e \in R_t^0$ if $t > t(e)$, and it follows that $t \mapsto \sum_{e \in R_t^0} h(e)$ is non-decreasing. Analogously, we can show that $t \mapsto \sum_{e \in L_t^2} h(e)$ is non-decreasing, while $t \mapsto \sum_{e \in R_t^2} h(e)$ is non-increasing. It follows that $\omega'(t)$ is non-decreasing, and so $\omega(t)$ is convex, proving the theorem. $\square$

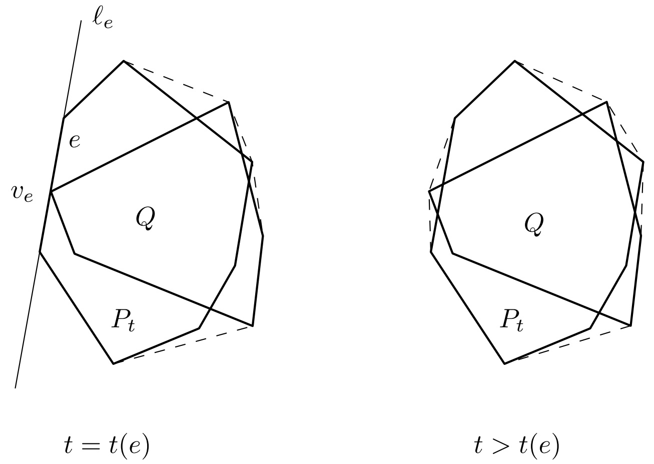


Figure 1: The unique *edge event* at time $t = t(e)$ for a left edge e.

4 Minimizing the convex hull under translations

Since the functions ψ and ω (for $k = 2$) are convex functions, they have a single local minimum. If the convex figures are polygons, then it is easy to evaluate ψ or ω for any $r \in \mathbb{R}^{2k}$, and so numerical optimization methods can be used to find the unique minimum.

In this section, we study the theoretical complexity of minimizing perimeter or area of the convex hull for *two convex polygons* P and Q.

Lemma 4. *Let P and Q be convex polygons with n vertices in total, and let ℓ be a line. Then we can compute a point $r \in \ell$ minimizing $|(P+r) \cup Q|$ in time $O(n \log n)$, and a point minimizing $||(P+r) \cup Q||$ in time $O(n)$.*

Proof. We choose a coordinate-system such that ℓ is the x-axis, and end up with the setting of the proof of Theorem 3. The combinatorial structure of $C_t = \mathrm{conv}(P_t \cup Q)$ changes only when a vertex of Q lies on a supporting line of P_t, and vice versa. There are $O(n)$ such values of t called breakpoints, and they can be computed in linear time: the vertex of Q hitting a line ℓ_e first must have a tangent parallel to ℓ_e, and so it suffices to merge the lists of slopes of P and Q. We sort all these values of t in time $O(n \log n)$.

For a fixed value of t, we can compute $\mathrm{conv}(P_t \cup Q)$ in linear time, since we know the sorted order of the vertices of P and Q in y-direction. We can therefore perform binary search on the list of breakpoints, and in $O(n \log n)$ time find an interval containing the optimum and at most one breakpoint. For the area function, the optimum must occur at a breakpoint, and we are done. For the perimeter function, it remains to find the optimum between the two breakpoints. This amounts to minimizing a sum of square roots, and it seems that this has to be done numerically.

For the area function, we can do better than this (at least asymptotically) by using a decimation approach. The details are similar to the algorithm described by Ahn et al. [2].

Let $\omega(t) = ||P_t \cup Q||$. As we saw in Theorem 3, $\omega'(t)$ can be expressed as a linear combination of $h(e)$ over all edges of P_t and Q, and changes only at a breakpoint. Let us define $\omega^-(t) := \lim_{\varepsilon \to 0} \omega'(t-\varepsilon)$ and $\omega^+(t) := \lim_{\varepsilon \to 0} \omega'(t+\varepsilon)$. Our task is then to find a breakpoint $t^* \in \mathbb{R}$ such that $\omega^-(t^*) \leqslant 0$ and $\omega^+(t^*) \geqslant 0$.

At each stage, we maintain an open interval (t_0, t_1), the values $\omega^+(t_0)$ and $\omega^-(t_1)$, and an *unordered* list of all breakpoints occurring in the open interval (t_0, t_1). We initialize the recursion by letting $(t_0, t_1) = (-\infty, \infty)$. As we observed above, we can compute the unordered list $\mathcal{L}$ of all breakpoints occurring between t_0 and t_1 in linear time. In a recursive step, we compute a median value t_2 for $\mathcal{L}$ and compute $\omega^-(t_2)$ and $\omega^+(t_2)$ by scanning $\mathcal{L}$, which takes time linear in the size of $\mathcal{L}$. If $\omega^-(t_2) \leqslant 0$ and $\omega^+(t_2) \geqslant 0$, we are done, and return t_2 as t^*. If both are negative, we scan $\mathcal{L}$ to create a list of breakpoints occurring strictly between t_2 and t_1, and recurse on the interval (t_2, t_1). Otherwise, that is if both are positive, we similarly recurse on the interval (t_0, t_2). Since the size of $\mathcal{L}$ decreases to half its previous size in each recursion step, the overall running time of our algorithm is $O(n)$. $\square$

We now have all the tools we need to find the optimal placement.

Theorem 5. *Let P and Q be convex polygons with n vertices in total. Then we can compute a translation $r \in \mathbb{R}^2$ minimizing $|(P+r) \cup Q|$ in time $O(n \log^2 n)$, and a translation minimizing $||(P+r) \cup Q||$ in time $O(n \log n)$.*

Proof. De Berg et al. [9] give an $O(n \log n)$ time algorithm to compute the translation r maximizing $||(P+r) \cap Q||$. Their algorithm makes use of a subroutine to find the best translation restricted to a line, and exploits the fact that the function $r \mapsto ||(P+r) \cap Q||$ is unimodal.

We have established the same property for the functions $r \mapsto |(P+r) \cup Q|$ and $r \mapsto ||(P+r) \cup Q||$ in Theorems 2 and 3. We have also given a subroutine to compute the optimum along a line in Lemma 4. We can therefore apply de Berg et al.'s technique to our problem. Their Lemmas 3.4 to 3.7 apply with minor modifications that we leave to the reader. $\square$

5 Minimizing the convex hull under rigid motions

We now consider the problem when arbitrary rigid motions of P are allowed, and give an approximation algorithm. Let $\xi(S)$ be the target function we wish to minimize, that is, either $\xi(S) = |S|$ or $\xi(S) = ||S||$. Let φ^{opt} be a rigid motion that minimizes the desired target function $\xi(\varphi^{\text{opt}}(S))$. For a given $\varepsilon > 0$, our goal will be to find a rigid motion φ^{app} such that $\xi(\varphi^{\text{app}} P \cup Q) \leqslant (1+\varepsilon) \cdot \xi(\varphi^{\text{opt}} P \cup Q)$.

Our algorithm is quite simple: We first generate a set D_ε of $O(1/\varepsilon)$ orientations of P. For each orientation, we then compute the optimal translation of the rotated copy of P, using Theorem 5. The total running time is clearly $O((1/\varepsilon) n \log n)$ for the area function and $O((1/\varepsilon) n \log^2 n)$ for the perimeter function, it remains to describe how to find D_ε and to prove the approximation bound.

For the perimeter function, we can simply sample orientations uniformly. For the area function, this only works when P and Q are rather round and "fat," but the sampling resolution would have to be too fine when they are long and skinny. Fortunately, in the latter case we can prove that the diameters of P and Q must be nearly aligned, and it suffices to sample very finely around the orientation that achieves alignment.

We first show a lower bound on $|P \cup Q|$ and $||P \cup Q||$ for two convex sets P and Q based on their diameter and width.

Lemma 6. *Let P and Q be convex sets in the plane. Then*

$$|P \cup Q| \geqslant 2 \max\{d(P), d(Q)\}$$
$$||P \cup Q|| \geqslant \frac{1}{2} \cdot \max\{d(P), d(Q)\} \cdot \max\{w(P), w(Q)\}$$

Proof. Let pq be a diameter of P, and assume that $d(P) \geqslant d(Q)$. Since pq is contained in $\operatorname{conv}(P \cup Q)$, the first inequality follows.

Let R be a rectangle circumscribed to P with two sides parallel to pq such that P touches all four sides of R at points p, q, r and s, and let w be the side of R orthogonal to pq.

Clearly, $\operatorname{conv}(p, q, r, s)$ is contained in $\operatorname{conv}(P \cup Q)$ and consists of two triangles with common base pq. Since $w \geqslant w(P)$, it has area at least $d(P) \cdot w(P)/2$. Now let $p'q'$ be a line segment in Q which has length $w(Q)$ and is orthogonal to pq. Note that there always exists such a segment: in fact, Q contains a segment of length $w(Q)$ of every direction [14, pg. 12, ex.4(a)]. Then $\operatorname{conv}(p, q, p', q')$ is contained in $\operatorname{conv}(P \cup Q)$, and it has area at least $d(P) \cdot w(Q)/2$. Therefore,

$$||P \cup Q|| \geqslant d(P) \cdot \frac{1}{2} \cdot \max\{w(P), w(Q)\},$$

and the second inequality follows. □

5.1 Minimizing the perimeter

We now prove that sampling orientations works for the perimeter function.

Lemma 7. *Let P and Q be convex sets in the plane, let $\varepsilon > 0$, and let ρ be a rotation of angle $\delta \leqslant \varepsilon/\pi$ around a point in P. Then $|\rho P \cup Q| \leqslant (1 + \varepsilon)|P \cup Q|$.*

Proof. Let $S = \operatorname{conv}(P \cup Q)$ and let $S' = \operatorname{conv}(\rho P \cup Q)$. We note that any point q in $S' \setminus S$ is at distance at most $\delta d(P)$ from the boundary of S. Let T denote the set of points that are at distance at most $\delta d(P)$ from S. Then we have $S' \subset T$. Using Eq. (2), we can bound the perimeter of S' as

$$|S'| \leqslant |T| = |S| + 2\pi\delta d(P) \leqslant |S| + 2\varepsilon d(P) \leqslant (1 + \varepsilon)|S|,$$

since by Lemma 6 we have $|S| \geqslant 2d(P)$. □

The algorithmic result is the following.

Lemma 8. *Let P and Q be convex polygons with n vertices in total, and let $\varepsilon > 0$. Then we can compute a rigid motion φ^{app} such that $|\varphi^{\mathrm{app}} P \cup Q| \leqslant (1 + \varepsilon)|\varphi^{\mathrm{opt}} P \cup Q|$ in time $O((1/\varepsilon)n \log^2 n)$.*

Proof. We sample orientations at interval ε/π. For each of these, we then compute the translation minimizing $|\varphi P \cup Q|$ using Theorem 5. We return the rigid motion achieving the minimum perimeter among the sampled orientations. The approximation bound follows from Lemma 7. Clearly we have sampled $O(1/\varepsilon)$ directions, and the whole procedure takes time $O((1/\varepsilon)n \log^2 n)$. □

5.2 Minimizing the area

For the area function, we cannot simply sample uniformly when the two objects are long and skinny. We now prove that in this case their diameters need to be nearly aligned.

Lemma 9. *Let P and Q be convex sets in the plane, let φ^{opt} be a rigid motion minimizing $||\varphi^{\mathrm{opt}} P \cup Q||$, and let γ be the smaller angle between two diameters of $\varphi^{\mathrm{opt}} P$ and Q. Then*

$$\sin\gamma \leqslant \frac{2 \cdot \max\{w(P), w(Q)\}}{\min\{d(P), d(Q)\}}$$

Proof. Let pq be a diameter of $\varphi^{\mathrm{opt}} P$ and let $p'q'$ be a diameter of Q. Assume that the diameters pq and $p'q'$ make an angle $\gamma \in [0, \pi/2]$, as in Fig. 2. If the two diameters intersect in a point x, then the convex hull consists of two triangles with common base pq and area

$$||\operatorname{conv}(p, q, p', q')|| = \frac{1}{2} \cdot d(P) \cdot (|p'x| + |xq'|) \sin\gamma = \frac{1}{2} \cdot d(P) \cdot d(Q) \cdot \sin\gamma.$$

If the two diameters do not intersect, we can always translate one of them until they intersect while the area function is non-increasing. Since $\operatorname{conv}(p, q, p', q') \subset \operatorname{conv}(\varphi^{\mathrm{opt}} P \cup Q)$, we have $||\varphi^{\mathrm{opt}} P \cup Q|| \geqslant \frac{1}{2} \cdot d(P) \cdot d(Q) \cdot \sin\gamma$.

Consider now rectangles R and R' circumscribed to P and Q, with R having side lengths $d(P)$ and $w(P)$, and R' having side lengths $d(Q)$ and $w(Q)$. There is a rigid motion φ such that $\varphi R \cup R'$ fits in a rectangle with sides $\max\{d(P), d(Q)\}$ and $\max\{w(P), w(Q)\}$, and so

$$||\varphi^{\text{opt}} P \cup Q|| \leqslant ||\varphi P \cup Q|| \leqslant ||\varphi R \cup R'|| \leqslant \max\{d(P), d(Q)\} \cdot \max\{w(P), w(Q)\}.$$

Combining the upper and lower bounds for $||\varphi^{\text{opt}} P \cup Q||$ proves the lemma. □

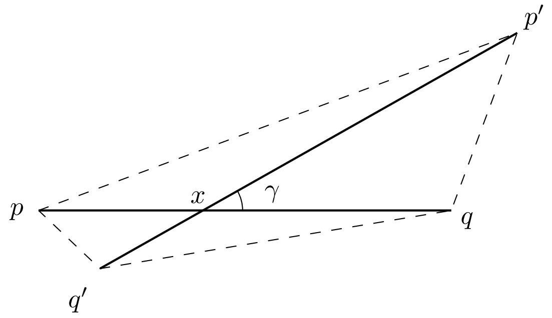


Figure 2: The convex hull of two diameters has area at least $\frac{1}{2} \cdot d(P) \cdot d(Q) \cdot \sin \gamma$.

We now prove that sampling orientations works.

Lemma 10. *Let P and Q be convex sets in the plane such that $P \cap Q \neq \emptyset$, let $\varepsilon > 0$, and let ρ be a rotation of angle*

$$\delta \leqslant \min \left\{ \frac{\varepsilon}{8} \frac{\max\{w(P), w(Q)\}}{\min\{d(P), d(Q)\}}, \frac{\pi}{12} \right\}$$

around a point in P. Then $||\rho P \cup Q|| \leqslant (1 + \varepsilon)||P \cup Q||$.

Proof. Without loss of generality, we assume that $d(P) \leqslant d(Q)$. Let S, S', and T as in Lemma 7. We have $S' \subset T$, and by Eq. (3) the area of T is

$$\begin{aligned}
||S'|| \leqslant ||T|| &= ||S|| + \delta d(P) \cdot |Q| + \pi(\delta d(P))^2 \\
&\leqslant ||S|| + \delta d(P) \cdot \pi d(Q) + \pi(\delta d(P))^2 \\
&\leqslant ||S|| + \delta d(P) \cdot \pi d(Q) + \pi \delta^2 d(P) \cdot d(Q) \\
&\leqslant ||S|| + \delta d(P) \cdot \pi d(Q) + \delta \frac{\pi^2}{12} d(P) \cdot d(Q) \\
&= ||S|| + \delta d(P) \cdot d(Q) \cdot (\pi + \frac{\pi^2}{12}) \\
&< ||S|| + 4\delta d(P) \cdot d(Q) \\
&\leqslant ||S|| + \frac{4\varepsilon}{8} \frac{d(P) \cdot d(Q)}{\min\{d(P), d(Q)\}} \max\{w(P), w(Q)\} \\
&= ||S|| + \frac{\varepsilon}{2} \max\{d(P), d(Q)\} \max\{w(P), w(Q)\} \\
&\leqslant (1 + \varepsilon)||S||.
\end{aligned}$$

(The last inequality follows from Lemma 6.) □

We can now state the algorithmic result.

Lemma 11. *Let P and Q be convex polygons with n vertices in total, and let $\varepsilon > 0$. Then we can compute a rigid motion φ^{app} such that $||\varphi^{\text{app}} P \cup Q|| \leqslant (1 + \varepsilon)||\varphi^{\text{opt}} P \cup Q||$ in time $O((1/\varepsilon)n \log n)$.*

Proof. In linear time, we compute width and diameter of both polygons. Let $L = \frac{\max\{w(P), w(Q)\}}{\min\{d(P), d(Q)\}}$. We sample orientations at interval $\varepsilon L/8$, but omitting all directions where the computed diameters make an angle γ with $\sin \gamma > 2L$. Clearly we have sampled $O(1/\varepsilon)$ directions. For each of these, we then compute the translation minimizing $||\varphi P \cup Q||$ using Theorem 5. The whole procedure takes time $O((1/\varepsilon)n \log n)$, and the approximation bound follows from Lemmas 9 and 10. □

5.3 Improving the running time

The time bounds in Lemmas 8 and 11 can be improved by a classic idea: we first replace the input by approximations with $O(1/\sqrt{\varepsilon})$ vertices (that is, the complexity of the problem is now independent of n). We then apply Lemma 8 or Lemma 11 to the approximations. This will also allow us to solve the problem for more general convex sets (not necessarily polygons).

Theorem 12. *Given two convex sets P and Q in the plane and $\varepsilon > 0$, we can compute a rigid motion φ^{app} such that $\xi(\varphi^{\text{app}} P \cup Q) \leqslant (1+\varepsilon) \min_\varphi \xi(\varphi P \cup Q)$, where the minimum is over all rigid motions. The running time is $O((T_P + T_Q)\varepsilon^{-1/2} + \varepsilon^{-3/2} \log^a(1/\varepsilon))$, where $a = 1$ if $\xi(S) = ||S||$ and $a = 2$ if $\xi(S) = |S|$.*

Proof. Let $\varepsilon' := \varepsilon/(3c_2)$, where $c_2 > 1$ is the constant needed for Ineq. (5). Using Lemma 1, we construct outer ε'-approximations P' and Q' to P and Q in time $O((T_P + T_Q)\varepsilon^{-1/2})$. We then apply Lemma 8 or Lemma 11 to compute a rigid motion φ^{app} such that $\xi(\varphi^{\text{app}} P' \cup Q') \leqslant (1+\varepsilon/3) \min_\varphi \xi(\varphi P' \cup Q')$. This takes time $O((\varepsilon^{-3/2}) \log^a(\varepsilon^{-1/2}))$, and it remains to prove the approximation bound.

By Lemma 1, P is an ε'-kernel for P', and Q is an ε'-kernel for Q'. This implies that $\text{conv}(\varphi^{\text{opt}} P \cup Q)$ is an ε'-kernel of $\text{conv}(\varphi^{\text{opt}} P' \cup Q')$, where φ^{opt} is a rigid motion minimizing $\xi(\varphi^{\text{opt}} P \cup Q)$, and so $(1 - \frac{\varepsilon}{3})\xi(\varphi^{\text{opt}} P' \cup Q') \leqslant \xi(\varphi^{\text{opt}} P \cup Q)$ by (4) and (5). We get

$$\begin{aligned}
\xi(\varphi^{\text{app}} P \cup Q) \leqslant \xi(\varphi^{\text{app}} P' \cup Q') &\leqslant (1 + \frac{\varepsilon}{3}) \min_\varphi \xi(\varphi P' \cup Q') \\
&\leqslant (1 + \frac{\varepsilon}{3})\xi(\varphi^{\text{opt}} P' \cup Q') \leqslant \frac{1 + \frac{\varepsilon}{3}}{1 - \frac{\varepsilon}{3}} \xi(\varphi^{\text{opt}} P \cup Q) \\
&\leqslant (1 + \varepsilon)\xi(\varphi^{\text{opt}} P \cup Q),
\end{aligned}$$

proving the approximation bound. □

6 When overlap is not allowed

In this section we briefly consider the problem when the two figures are not allowed to overlap. That is, we are given two convex polygons P and Q with n vertices in total, and we wish to minimize either the perimeter or the area of the convex hull under the restriction that the interiors of φP and Q remain disjoint.

For the case of translations only, Lee and Woo [13] presented a linear time algorithm for finding a translation that minimizes the area of the convex hull. The idea of the algorithm is to "slide" P around Q, keeping their boundaries in contact, while maintaining the area of the convex hull. Since there is only a linear number of changes to the combinatorial structure of the convex hull, this can be done in linear time. While not mentioned in the paper, the same algorithm works for minimizing the perimeter of the convex hull.

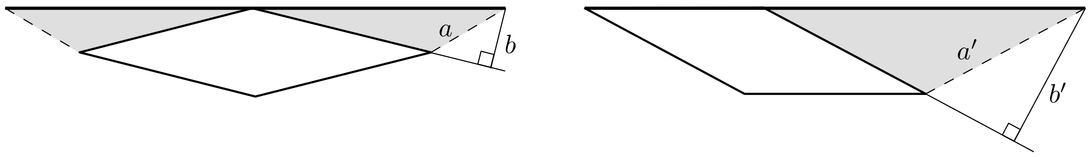
It remains to find the best *rigid motion* for this problem. One may conjecture that the optimal solution is obtained when two edges are in contact. This is, however, not always the case (see Fig. 3). As mentioned earlier, Tang et al. [20] gave an $O(n^3)$ time deterministic algorithm for finding a rigid motion that minimizes the area of the convex hull *exactly*. In contrast, we present a near-linear time *approximation* algorithm for finding a rigid motion that minimizes either the perimeter or the area of their convex hull.

We proceed very similarly to Section 5. For the area case, we need the following technical lemma.

Lemma 13. *Let P and Q be convex sets in the plane, let φ^{opt} be a rigid motion minimizing $||\varphi^{\text{opt}} P \cup Q||$ while keeping their interiors disjoint, and let γ be the smaller angle between two diameters of $\varphi^{\text{opt}} P$ and Q. Then,*

$$\sin\gamma \leqslant \frac{4 \cdot \max\{w(P), w(Q)\}}{\min\{d(P), d(Q)\}}$$

Proof. As in Lemma 9, we first argue that $|\varphi^{\text{opt}} P \cup Q| \geqslant \frac{1}{2} \cdot d(P) \cdot d(Q) \cdot \sin\gamma$ (the restriction can only increase the optimum). We then again consider the rectangles R and R' (as defined in the proof of Lemma 9). There is a rigid motion φ that aligns them such that $\varphi R \cup R'$ fits in a rectangle with sides $2 \cdot \max\{d(P), d(Q)\}$ and $\max\{w(P), w(Q)\}$, implying the bound. □



(a) (b)

Figure 3: (a) An example consisting of a long and very thin rectangle and a rhombus touching the rectangle from below with edge-vertex contact. (b) A configuration with edge-edge contact; we can always adjust the internal angles of the rhombus and the horizontal length of the rectangle such that $a' > 2a$ and $b' > 2b$ as in the figure. Clearly, the convex hull of the two figures has larger perimeter than their convex hull in configuration (a), and the grey triangle has larger area than the two grey triangles in (a). Moreover, it is not difficult to see that no horizontal translation of the rhombus can reduce the perimeter or the area the convex hull.

Theorem 14. *Given two convex sets P and Q in the plane and $\varepsilon > 0$, we can compute in time $O((T_P + T_Q)\varepsilon^{-1/2} + \varepsilon^{-3/2})$ a rigid motion φ^{app} such that $\xi(\varphi^{\mathrm{app}}P \cup Q) \leqslant (1+\varepsilon)\min_\varphi \xi(\varphi P \cup Q)$ over all rigid motions, where either $\xi(S) = |S|$ or $\xi(S) = ||S||$.*

Proof. Let us first assume that P and Q are convex n-gons. We sample $O(1/\varepsilon)$ orientations as follows: For the perimeter case, we sample orientations at interval ε/π. For the area case, we let $L = \frac{4\cdot\max\{w(P),w(Q)\}}{\min\{d(P),d(Q)\}}$, and we sample orientations at interval $\varepsilon L/8$, but omitting all orientations where two diameters of P and Q make an angle γ with $\sin\gamma > 4L$. In both cases, we have sampled $O(1/\varepsilon)$ orientations. For each sampled orientation, we compute the translation minimizing $\xi((P + r) \cup Q)$ under the restriction $\mathrm{int}(P + r) \cap \mathrm{int}(Q) = \emptyset$ using the $O(n)$ time algorithm of Lee and Woo [13].

This procedure takes time $O(n/\varepsilon)$ and results in a rigid motion φ^{app} with $\xi(\varphi^{\mathrm{app}}P \cup Q) \leqslant (1+\varepsilon)\min_\varphi \xi(\varphi P \cup Q)$. This follows from Lemmas 7, 10, and 13.

Finally, we improve the running time exactly as in Theorem 12: We construct outer approximations P' and Q' of P and Q in time $O((T_P + T_Q)\varepsilon^{-1/2})$, and then apply the procedure described above in time $O(\varepsilon^{-1/2}/\varepsilon) = O(\varepsilon^{-3/2})$. ☐

Acknowledgment

We acknowledge helpful discussions with Takeshi Tokuyama.

References

[1] P. K. Agarwal, S. Har-Peled, K. R. Varadarajan. Approximating extent measures of points. *Journal of the ACM*, 51:606–635, 2004.

[2] H.-K. Ahn, P. Brass, O. Cheong, H.-S. Na, C.-S. Shin, and A. Vigneron. Inscribing an axially symmetric polygon and other approximation algorithms for planar convex sets. *Computational Geometry: Theory and Applications*, 33:152–164, 2006.

[3] H.-K. Ahn, P. Brass, C.-S. Shin. Maximum overlap and minimum convex hull of two convex polyhedra under translations. *Computational Geometry: Theory and Applications*, 40:171–177, 2008.

[4] H.-K. Ahn, O. Cheong, C.-D. Park, C.-S. Shin, and A. Vigneron. Maximizing the overlap of two planar convex sets under rigid motions. *Computational Geometry: Theory and Applications*, 37:3–15, 2007.

[5] H. Alt, J. Blömer, M. Godau, and H. Wagener. Approximation of convex polygons. In *Proc. 17th International Colloquium on Automata, Languages and Programming*, LNCS 443, pages 703–716, Springer-Verlag 1990.

[6] H. Alt, U. Fuchs, G. Rote, and G. Weber. Matching convex shapes with respect to the symmetric difference. *Algorithmica*, 21:89–103, 1998.

[7] H. Alt, F. Hurtado. Packing convex polygons into rectangular boxes. In *Proc. Japanese Conference on Discrete and Computational Geometry 2000*, LNCS 2098, pages 67–80, Springer-Verlag 2001.

[8] M. de Berg, S. Cabello, P. Giannopoulos, C. Knauer, R. van Oostrum, and R. C. Veltkamp. Maximizing the area of overlap of two unions of disks under rigid motion. In *Proc. Scandinavian Workshop on Algorithm Theory*, pages 138–149, 2004.

[9] M. de Berg, O. Cheong, O. Devillers, M. van Kreveld, and M. Teillaud. Computing the maximum overlap of two convex polygons under translations. *Theory of Computing Systems*, 31:613–628, 1998.

[10] M. do Carmo. *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[11] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *J. Approximation Theory*, 10:227–236, 1974; Erratum in *J. Approximation Theory*, 26:192–193, 1979.

[12] J. Egeblad, B. K. Nielsen, and M. Brazil. Translational packing of arbitrary polytopes. *Computational Geometry: Theory and Applications*, 42:269–288, 2009.

[13] H. C. Lee and T. C. Woo. Determining in linear time the minimum area convex hull of two polygons. *IIE Transactions*, 20:338–345, 1988.

[14] J. Matoušek. *Lectures on Discrete Geometry*, Springer-Verlag, 2002.

[15] V. J. Milenkovic. Rotational polygon containment and minimum enclosure. In *Proc. 14th Annual ACM Symposium on Computational Geometry*, pages 1–8, 1998.

[16] V. J. Milenkovic. Rotational polygon containment and minimum enclosure using only robust 2D constructions. *Computational Geometry: Theory and Applications*, 13:3–19, 1999.

[17] V. J. Milenkovic. Translational polygon containment and minimum enclosure using linear programming based restriction. In *Proc. 28th Annual ACM Symposium on Theory of Computation*, pages 109–118, 1996.

[18] D. M. Mount, R. Silverman, and A. Y. Wu. On the area of overlap of translated polygons. *Computer Vision and Image Understanding: CVIU*, 64:53–61, 1996.

[19] K. Sugihara, M. Sawai, H. Sano, D.-S. Kim and D. Kim. Disk packing for the estimation of the size of a wire bundle. *Japan Journal of Industrial and Applied Mathematics*, 21:259–278, 2004.

[20] K. Tang, C. C. L. Wang, D. Z. Chen. Minimum area convex packing of two convex polygons. *International Journal of Computational Geometry and Applications*, 16:41–74, 2006.

[21] I. M. Yaglom and V. G. Boltyanskii. *Convex figures*. Holt, Rinehart and Winston, New York, 1961.