Minimum-Width Annulus with Outliers: Circular, Square, and Rectangular Cases*

Hee-Kap Ahn† Taehoon Ahn† Sang Won Bae‡ Jongmin Choi†
Mincheol Kim† Eunjin Oh§ Chan-Su Shin¶ Sang Duk Yoon‖

Abstract

We study the problem of computing a minimum-width annulus with outliers. Specifically, given a set of n points in the plane and an integer k with $1 \leq k \leq n$, the problem asks to find a minimum-width annulus that contains at least $n-k$ input points. The k excluded points are considered as outliers of the input points. In this paper, we are interested in particular in annuli of three different shapes: circular, square, and rectangular annuli. For the three cases, we present first and improved algorithms to the problem.

1 Introduction

An annulus is a region bounded by two concentric circles. There are a few applications of computing the minimum annulus enclosing a set of points in the plane. For instance, one of the topics in metrology is to measure the roundness of an object, which is done mostly by measuring points obtained from the boundary of the object. If the width of the annulus that covers the measured points is close to zero or below a predefined threshold, then one can say that the object is (almost) round. Otherwise, the object is not round enough, and therefore it should be rejected. Another application is to locate an obnoxious or undesirable facility for a set of sites that use or get served by the facility in the plane. No one wants to have an obnoxious facility such as a garbage dump in his/her backyard but it should be located within a reasonable distance from the sites. A good location for such an obnoxious facility is the one whose closest site is far enough and whose farthest site is not too far.

The *minimum-width* annulus that encloses a set P of points reflects the roundness of the point set P well. There has been a fair amount of work on the minimum-width annulus for a set of points in the plane. However, the data we obtain in applications often contains outliers which are due to variability in the measurement or errors in transmission. Outliers can be seen as violation of constraints in the minimum-width annulus problem: the points of P are to be

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†Department of Computer Science and Engineering, POSTECH, South Korea. Emails: {heekap, sloth, icothos, rucatia}@postech.ac.kr.

‡Division of Computer Science and Engineering, Kyonggi University, South Korea. Email: swbae@kgu.ac.kr.

§Max-Planck-Institut für Informatik, Germany. Email: eoh@mpi-inf.mpg.de.

¶Division of Computer and Electronic Systems Engineering, Hankuk University of Foreign Studies, South Korea. Email: cssin@hufs.ac.kr.

‖Samsung Display Co. Ltd., South Korea. Email: sangduk.yoon@samsung.com.

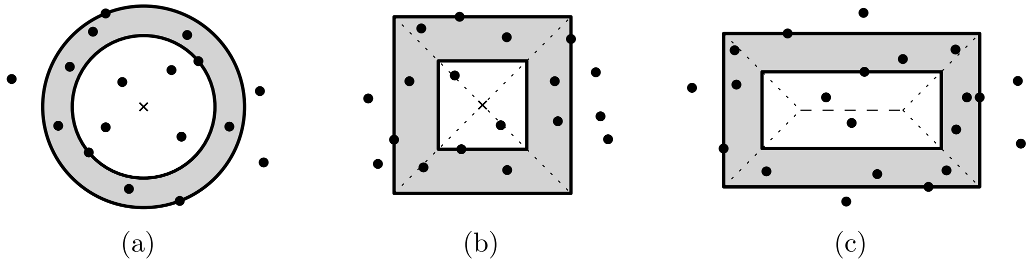

Figure 1: Minimum-width (a) circular, (b) square, and (c) rectangular annuli with $k = 7$ outliers for a given set of points.

covered by the annulus but some of them are allowed to be violated. In this paper, we study the minimum-width annulus problem for points allowing outliers in the plane.

The minimum-width annulus problem has been studied in computational geometry. Motivated by the roundness test in metrology, Ebara et al. [9] presented a simple quadratic time algorithm that computes a minimum-width circular annulus enclosing a given set of n points in the plane using Voronoi diagrams. Later, Agarwal, Sharir, and Toledo [3] presented an algorithm that uses Megiddo's parametric search technique and computes the minimum-width circular annulus in $O(n^{8/5+\varepsilon})$ time for any $\varepsilon > 0$. The problem was reconsidered by Agarwal et al. [4] as an application of computing the vertices, edges and 2-dimensional faces of the lower envelopes of multivariate functions. Their algorithm takes $O(n^{17/11+\varepsilon})$ expected time for any $\varepsilon > 0$. Then, Agarwal and Sharir [2] simplified and improved their previous algorithm by using vertical decomposition to $O(n^{3/2+\varepsilon})$ expected time for any $\varepsilon > 0$. Chan gave an $(1 + \varepsilon)$-approximation algorithm for the problem [7]. However, there is no exact algorithm known for the problem in the presence of outliers, except an approximation algorithm by Har-Peled and Wang [11].

There also has been work on variations of the minimum-width annulus problem in the plane, depending on the *shape* of the annulus as well as the *distance metric* for measuring the width. A square or rectangular annulus is the region bounded by two concentric axis-parallel squares or rectangles, respectively. Abellanas et al. [1] presented an $O(n)$-time algorithm for the rectangular annulus problem and considered several variations of the problem. Gluchshenko, Hamacher, and Tamir [10] gave an optimal $O(n \log n)$-time algorithm for a minimum-width square annulus. Later, Mukherjee et al. [14] presented an $O(n^2 \log n)$-time algorithm that computes a minimum-width rectangular annulus over all orientations, and Bae [6] showed that a minimum-width square annulus over all orientations can be computed in $O(n^3 \log n)$ time.

In this paper, we study the problem of computing a minimum-width annulus that contains at least $n - k$ input points, when n points are given as input points and $1 \leq k \leq n$ is also a part of input. Since the input points not covered by the resulting annulus are considered as the *outliers* of the annulus, this problem is also called as the minimum-width annulus problem with k outliers. We are interested in annuli of three different shapes: circular, square and rectangular annuli. See Figure 1 for an illustration.

Very recently, Bae [5] considered the square or rectangular annulus problem with $k \geq 1$ outliers and presented several first algorithms. Among them, he presented an $O(k^2 n \log n + k^3 n)$-time algorithm for the square annulus with k outliers and $O(nk^2 \log k + k^4 \log^3 k)$-time algorithm for the rectangular annulus with k outliers. It is worth noting that, when k is a constant, these running times match the lower bounds of the problems. On the other hand, no nontrivial algorithm for the minimum-width circular annulus with outliers is known so far.

Our results in this paper are threefold:

- We give an $O(k(kn)^{3/2+\varepsilon})$-time algorithm for the minimum-width circular annulus with k outliers. This is the first nontrivial algorithm for the circular variant of the problem.
- We present an $O(k^2 n \log n)$-time algorithm for the minimum-width square annulus problem with k outliers. This improves the previously best algorithm by Bae [5], which takes $O(k^2 n \log n + k^3 n)$ time.
- We also present two algorithms for the minimum-width rectangular annulus with k outliers whose running times are $O(n \log n + k^4 \log^2 n)$ and $O(nk^2 \log k + k^4 \log^2 k)$. Both of our algorithms are faster than the previously best known ones [5], which take $O(n \log^2 n + k^4 \log k \log^2 n)$ time and $O(nk^2 \log k + k^4 \log^3 k)$ time, respectively.

2 Preliminaries

In this paper, we are interested in annuli of three different shapes: circular, square, and rectangular annuli. A *circular annulus* is a closed region in the plane bounded by two concentric circles. The bigger circle that bounds a circular annulus A is called the *outer circle* of A, while the other is called the *inner circle* of A. The *width* of a circular annulus is the difference between the radii of its outer and inner circles.

For the square and rectangular cases, we consider only axis-parallel squares and rectangles. So, throughout the paper, any square or rectangle we discuss is supposed to be axis-parallel, unless stated otherwise. Consider a rectangle, or possibly a square, R in the plane $\mathbb{R}^2$. We call the intersection point of its two diagonals the *center* of R.

An *(inward) offset* of R by $\delta > 0$ is a rectangle obtained by sliding the four sides of R inwards by δ. If the shorter side of R is of length r, then the offset of R by $\delta = \frac{1}{2}r$ is degenerated to a line segment or a point. For any positive $\delta \leq \frac{1}{2}r$, consider an inward offset R' of R by δ. Then, the closed region A between R and R', including its boundary, is called a *rectangular annulus* with the *outer rectangle* R and the *inner rectangle* R'. When R is a square and so is R', the annulus A is called a *square annulus*, and R and R' are called its *outer square* and *inner square*, respectively. The offset value δ is called the *width* of the annulus.

Consider an annulus A, regardless of its shape. The complement $\mathbb{R}^2 \setminus A$ of the annulus A consists of two connected components. We shall call the component enclosed by A the *inner part* of A and the other component the *outer part* of A.

Given a set P of n points in $\mathbb{R}^2$ and an integer k with $1 \leq k \leq n$, our problem asks a minimum-width circular, square, or rectangular annulus that contains at least $n - k$ points of P. The input points that are not covered by the resulting annulus are called *outliers.* We call such an annulus that contains at least $n - k$ point of P an *annulus of P with k outliers.* In the following sections, we address the problem of computing a minimum-width circular, square, and rectangular annuli separately, in this order.

3 Circular Annulus with Outliers

In this section, we consider the problem of finding a minimum-width circular annulus with k outliers for a given set P of n points. As observed in [2, 3, 4], the outer and inner circles of a minimum-width circular annulus (without outliers) are determined by four points of P: (1) one of its outer and inner circles has three points on it and the other has one, or (2) both have two points on each. This implies the following lemma for minimum-width circular annuli with outliers.

Lemma 1 *There exists a minimum-width circular annulus A of P with k outliers such that one of the following conditions holds:*

(1) Three points of P lie on one of the inner and outer circles of A, and one point in P lies on the other circle.

(2) Both the inner and outer circles of A have two points of P on each.

Proof. Let A be any minimum-width circular annulus of P with k outliers. Let $P' := P \cap A$ be the set of points in P that are contained in A. Note that $|P'| \geq n - k$ by the requirement of the problem.

Now, consider a minimum-width circular annulus A' (without outliers) that contains all points in P' and satisfies one of the conditions (1) and (2). Such an annulus A' exists as observed in [2, 3, 4]. On one hand, by definition, the width of A' is not larger than that of A. On the other hand, since $|P \cap A'| \geq |P' \cap A'| = |P'| \geq n - k$, A' is a circular annulus with at most k outliers. Therefore, A' is another minimum-width circular annulus of P with k outliers that satisfies the desired condition. □

Consider a minimum-width circular annulus A of P with k outliers that satisfies one of conditions (1) and (2) stated in Lemma 1. Let k_{in} be the number of points in P that lie in the inner part of A and $k_{\text{out}} = k - k_{\text{in}}$ be the number of points in P that lie in the outer part of A. In the following, we will show that the center of A is related to the *higher-order Voronoi diagrams* of P. The *order-t Voronoi diagram* of P, denoted by $\mathcal{V}_t(P)$, decomposes the plane $\mathbb{R}^2$ into Voronoi regions such that all points in each Voronoi region share the common t nearest points among those in P [12]. Note that the ordinary nearest-neighbor Voronoi diagram of P is the order-1 diagram $\mathcal{V}_1(P)$, while the farthest-neighbor Voronoi diagram of P is the order-$(n-1)$ diagram $\mathcal{V}_{n-1}(P)$. For more details on the order-t Voronoi diagrams, refer to Lee [12] and Liu et al. [13].

Back to our discussion on annulus A, first assume that the annulus A falls in case (1), so one of the inner and outer circles of A has three points on it. If the inner circle C_{in} of A has three points of P on it, then observe that the center of C_{in}, so the center of A, coincides with a vertex of the order-$(k_{\text{in}} + 1)$ Voronoi diagram of P, since C_{in} contains k_{in} points of P in its interior. If the outer circle C_{out} of A has three points of P on it, then the center of A coincides with a vertex of the order-$(n - k_{\text{out}} - 1)$ diagram of P since C_{out} encloses exactly $n - k_{\text{out}}$ points of P.

Next, we assume case (2), so both the inner and outer circles of A have two points of P on each. Since the inner circle C_{in} of A contains k_{in} points in its interior and two points on C_{in}, the center of C_{in} lies on an edge of the order-$(k_{\text{in}} + 1)$ Voronoi diagram of P. On the other hand, since the outer circle C_{out} of A encloses all but k_{out} points of P and two points of P lie on C_{out}, the center of C_{out} lies on an edge of the order-$(n - k_{\text{out}} - 1)$ diagram of P. This implies that the center of A coincides with the intersection point of two edges of the Voronoi diagrams, one edge from each diagram.

Lemma 2 *There exists a minimum-width circular annulus of P with k outliers such that its center lies on a vertex of the overlay of the order-$(k' + 1)$ Voronoi diagram and the order-$(n - 1 - k + k')$ Voronoi diagram of P for some $0 \leq k' \leq k$.*

This already yields a nontrivial algorithm: For each $0 \leq k' \leq k$, compute the two diagrams, compute the overlay of them, and check every vertex of the overlay. Since the diagram $\mathcal{V}_t(P)$ has complexity $O(t(n-t))$ and can be computed in $O(t(n-t)\log n)$ time [13], we can compute a minimum-width circular annulus of P with k outliers roughly in time $O(k^3 n^2)$. In the following, we give a better solution.

Again, consider two cases stated in Lemma 1. As discussed above, in case (1), the center of our annulus A lies on a vertex of a higher-order Voronoi diagram. Thus, solutions falling into this case can be found without computing the overlay. For the purpose, we compute the order-t diagrams of P for $t = 1, 2, \ldots, k+1$ and $t = n-1, n-2, \ldots, n-k-1$ in $O(k^2 n \log n)$ time [12, 13]. After processing each of the diagrams for a standard point location structure [8], we check for each vertex of the order-$(k'+1)$ Voronoi diagram its location on the order-$(n-1-k+k')$ Voronoi diagram, which takes $O(\log n)$ time per vertex. Hence, this case can be handled in total $O(k^2 n \log n)$ time.

Case (2) is relatively tricky. As above, consider a minimum-width circular annulus A of P with k outliers. Assume that this is case (2) and that k_{in} points of P lie in the inner part of A and $k_{\text{out}} = k - k_{\text{in}}$ points of P lie in the outer part of A. Here, we can directly extend the algorithm by Agarwal and Sharir [2] of computing the minimum-width annulus in $O(n^{3/2+\varepsilon})$ time as follows. In this case, as discussed above, each of the outer and inner circles passes through two points, thus its center lies on an edge of the corresponding higher-order Voronoi diagram. Using the lifting transformation that maps a point (p_x, p_y) of P in the xy-plane into the point $(p_x, p_y, p_x^2 + p_y^2)$ on the paraboloid $z = x^2 + y^2$ in three dimension, a circle C with center (a, b) in the xy-plane is mapped to a plane $H(C)$ which is parallel to the plane tangent to the paraboloid at point $(a, b, a^2 + b^2)$ in three dimension. The plane $H(C)$ intersects the paraboloid in a closed curve whose orthogonal projection onto the xy-plane coincides with C. Furthermore, a point lies on, inside, outside C in the xy-plane if and only if its lifted point is on, below, above $H(C)$ in three dimension, respectively. Thus, if our annulus A misses k_{in} points in its inner part and k_{out} points in its outer part, then it is mapped to a pair of parallel planes in three dimension such that k_{in} points lie below the plane mapped from the inner circle and k_{out} points above the plane mapped from the outer circle, that is, $n - k_i n - k_o ut$ points lie in the strip between the two parallel planes.

A minimum-width annulus in the xy-plane is transformed into a pair of two parallel planes in three dimension that minimizes a (properly defined) distance function between the planes under the lifting [2]. In case (2), we observe that each of the two parallel planes contains two lifted points, thus its projected circle has its center on the bisector of the two points, i.e., on an edge of the higher-order Voronoi diagram in the xy-plane. By the same argument of Agarwal and Sharir [2], the problem of finding two parallel planes, each containing a line connecting two (lifted) points, with a minimum distance can be reduced to the problem of computing a closest pair of bichromatic lines in three dimension. This type of the closest line-pair problem can be solved in $O((|U| + |L|)^{3/2+\varepsilon})$ expected time for any positive $\varepsilon > 0$ by a randomized divide-and-conquer algorithm [2], where U is a set of the candidate lines contained in the upper plane (pairs of points lying on the outer circle) and L is a set of the candidate lines contained in the lower plane (pairs of points lying on the inner circle). Since U is obtained from the edges of the order-$(n - k_{\text{out}} - 1)$ Voronoi diagram of P and L is obtained from the edges of the order-$(k_{\text{in}}+1)$ Voronoi diagram of P, we have $|U| = O(k_{\text{out}}(n - k_{\text{out}})) = O(kn)$ and $|L| = O(k_{\text{in}}(n - k_{\text{in}})) = O(kn)$. Thus, it takes $O((kn)^{3/2+\varepsilon})$ expected time.

In order to handle case (2), we compute all those higher-order Voronoi diagrams, and for each $0 \le k' \le k$, we invoke the above algorithm for the closest line-pair problem. This correctly finds a minimum-width circular annulus with k outliers in case (2), and takes in total $O(k(kn)^{3/2+\varepsilon})$ expected time. Hence, we conclude the following theorem.

Theorem 1 *Given a set P of n points in the plane and an integer k with $1 \le k \le n$, a minimum-width circular annulus of P with k outliers can be computed in $O(k(kn)^{3/2+\varepsilon})$ expected time for any $\varepsilon > 0$.*

4 Square Annulus with Outliers

In this section, we study the minimum-width square annulus problem with outliers, and present an $O(k^2 n \log n)$-time algorithm that computes a minimum-width square annulus of a set P of n points with k outliers. Throughout this section, for a point p, we denote by $x(p)$ and $y(p)$ the x-coordinate of p and the y-coordinate of p, respectively.

4.1 Configuration of optimal solutions

Bae [5] showed the following configuration of an optimal solution.

Lemma 3 (Bae [5]) *There exists a minimum-width square annulus of P with k outliers that contains two points of P lying on the opposite sides of its outer square.*

Moreover, the following lemma holds.

Lemma 4 *There exists a minimum-width square annulus of P with k outliers such that either*

(1) one side of its inner square contains a point in P and three sides of its outer square contain points of P, or

(2) two sides of its inner square contain points of P and two opposite sides of its outer square contain points of P.

Proof. Consider a minimum-width square annulus A of P with k outliers that satisfies the condition stated in Lemma 3. Without loss of generality, we assume that the left and right sides of the outer square S_{out} of A contain a point in P on each. In addition, it is clear that the inner square S_{in} of A also has at least one point $p_0 \in P$ on its boundary due to the minimality of the width of A.

Suppose that A satisfies none of conditions (1) and (2) stated in the lemma. Here, we describe how to transform A into another minimum-width square annulus A' of P with k outliers satisfying condition (1) or (2). There are two cases: (i) when the point p_0 lies on the left or right side of S_{in}, or (ii) on its top or bottom side.

Consider the former case (i). In this case, we can slide A locally in a vertical direction, downwards or upwards. We slide A upwards (a) until the top or bottom side of S_{out} hits a point in P or (b) until the top or bottom side of S_{in} hits a point in P. Let A' be the resulting square annulus obtained from the above transformation. If this is case (a), then three sides of the outer square of A' contains a point in P on each; if this is case (b), then two sides of the inner square of A' contains a point in P on each. Since A' contains all the points in $P \cap A$, A' is indeed another minimum-width square annulus of P with k outliers.

Next, we consider case (ii). Without loss of generality, we assume that p_0 lies on the top side of S_{in}. Then, there exists a sufficiently small positive real $\epsilon > 0$ such that the translate A' of A downwards by ϵ contains all the points in $P \cap A$, and no point in P lies on the boundary of the inner square of A'. This implies that we can enlarge the inner square of A' while containing all the points in $P \cap A'$. This is a contradiction to the minimality of the width of A. So, case (ii) cannot happen. $\square$

Consider a minimum-width square annulus of P with k outliers satisfying the condition in Lemma 4. We assume without loss of generality that both the left and right sides of its outer square contain points p_L and p_R in P, respectively. In the following, we describe how to find such an optimal solution, if any.

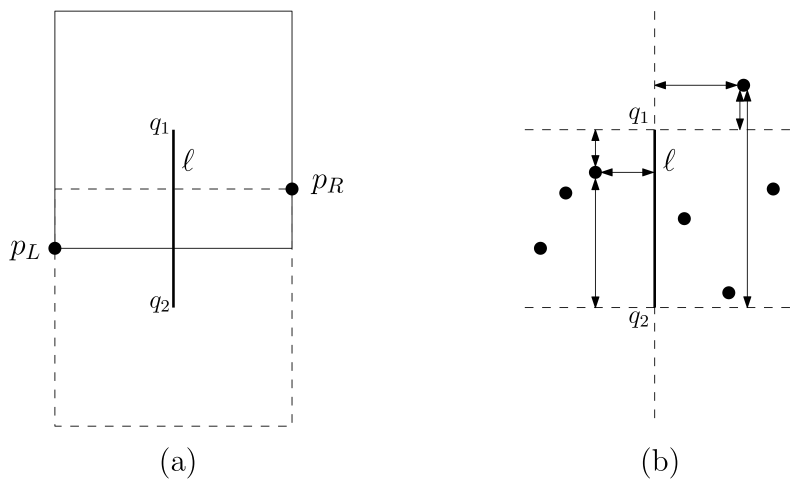

Figure 2: (a) The set of centers of the squares containing p_L and p_R on their left and right sides, respectively, form a vertical line segment $\ell = q_1q_2$. (b) Candidate side lengths of inner squares with respect to ℓ. For each point p, its x-distance to ℓ and y-distance to each of the endpoints q_1 and q_2 of ℓ are candidate side lengths of inner squares. Precisely, $2|x(p) - x(\ell)|$,$2|y(p) - y(q_1)|$, and $2|y(p) - y(q_2)|$ are candidate side lengths of inner squares for a point $p \in P$.

4.2 Finding candidate outer squares

We observe that there are at most k points of P lying to the left of p_L. Similarly, there are at most k points lying to the right of p_R. To use this observation, we compute the set $\mathcal{C}$ of pairs (p'_L, p'_R) of points in P such that there are at most k points in P lying to the left of p'_L and at most k points of P lying to the right of p'_R. Clearly, the size of $\mathcal{C}$ is $O(k^2)$ and (p_L, p_R) is contained in $\mathcal{C}$.

In the following subsection, we present an algorithm that computes a minimum-width square annulus of P with k outliers in $O(n \log n)$ time, provided we are given p_L and p_R. To obtain a minimum-width square annulus, we apply this procedure with each pair of $\mathcal{C}$. Then we obtain $O(k^2)$ square annuli one of which is an optimal solution. We simply choose the one with smallest width.

4.3 Finding the largest inner square for a candidate pair

Assume that we know p_L and p_R. We present an $O(n \log n)$-time algorithm for computing a minimum-width square annulus of P with k outliers such that p_L lies on the left side of its outer square and p_R lies on the right side of its outer square.

Consider any square whose left side contains p_L and whose right side contains p_R. Then it has the side length r_0, that is, $r_0 = x(p_R) - x(p_L)$, the difference of the x-coordinates of p_L and p_R. Moreover, its center lies on a vertical line segment ℓ that connects two points q_1 and q_2 such that: $x(q_1) = x(q_2) = x(p_L) + r_0/2$, $y(q_1) = \min\{y(p_L), y(p_R)\} + r_0/2$, and $y(q_2) = \max\{y(p_L), y(p_R)\} - r_0/2$. Furthermore, for any point $t \in \ell$, there is a square centered at t that contains p_L and p_R on its left and right sides, respectively. See Figure 2(a) for an illustration.

For any point $t \in \ell$, we denote the square centered at t with side length r_0 by $S_{\text{out}}(t)$. By definition, it contains p_L and p_R on its left and right sides, respectively. We use $S_{\text{in}}(t, r)$ to denote the inner square centered at t with side length r for $r \le r_0$.

In the following, we find a largest possible inner square for a candidate pair (p_L, p_R) on the outer square whose corresponding annulus contains at least $n-k$ points. That is, we maximize $r \in [0, r_0]$ such that a square annulus formed by $S_{\text{out}}(t)$ and $S_{\text{in}}(t, r)$ contains at least $n-k$ points of P for some $t \in \ell$. This determines the minimum-width square annulus for the fixed pair (p_L, p_R).

For the purpose, we compute the set of $O(n)$ candidate side lengths of inner square. We then apply a binary search on the sorted list L of the candidate side lengths using the decision algorithm to find the interval I of two consecutive side lengths in L containing the side length r^* of the largest inner square whose corresponding annulus contains at least $n-k$ points of P. To finally determine r^*, we apply a constant number of linear searches by scanning certain lists of selected points of P. Details will be given below.

4.3.1 Decision algorithm.

First, we describe our decision algorithm that tests whether $r \le r^*$ or not for a given side length $r \in [0, r_0]$. Assume that the points of P are sorted with respect to their y-coordinates. This allows us to sort the points in any subset P' of P with respect to their y-coordinates in $O(n)$ time. Here, r is given as input.

Imagine that a point t translates from one endpoint to the other endpoint of ℓ along ℓ. As t translates along ℓ, a point in P enters into $S_{\text{out}}(t)$ or $S_{\text{in}}(t, r)$, or exits from $S_{\text{out}}(t)$ or $S_{\text{in}}(t, r)$. We call a point $t \in \ell$ such that a point in P lies on the boundary of $S_{\text{out}}(t)$ or $S_{\text{in}}(t, r)$ an *event*.

Clearly, there are $O(n)$ such events, and we can compute and sort them along ℓ in $O(n)$ time since we maintain the sorted list of points in P in their y-coordinates. Imagine that we translate t along ℓ continuously. Any change in the number of points lying in the translated annulus occurs at an event. In other words, for a point t lying between any two consecutive events along ℓ, the set $P \cap S_{\text{out}}(t)$ and the set $P \cap S_{\text{in}}(t, r)$ remain the same. Thus, we compute the number of points of P lying in the annulus with outer square $S_{\text{out}}(t)$ and inner square $S_{\text{in}}(t, r)$ for every event in the sorted order by scanning the sorted list of events linearly and updating the number of points of P lying in the annulus in $O(1)$ time per event.

Hence, we can determine whether there is some $t \in \ell$ such that the number of points of P lying in the square annulus determined by $S_{\text{out}}(t)$ and $S_{\text{in}}(t, r)$ is at least $n-k$ in $O(n)$ time in total.

The above discussion is summarized into the following lemma.

Lemma 5 *Given $p_L, p_R \in P$ and $r > 0$, we can check in $O(n)$ time whether there is a square annulus of P with k outliers such that the outer square contains p_L and p_R on its the left side and on its right side, respectively, and its inner square has side length at least r for any input $r > 0$, provided that points in P are sorted with respect to their y-coordinates.*

4.3.2 Binary search on candidate side lengths.

We first compute $O(n)$ candidate side lengths of the largest inner square whose corresponding annulus contains $n-k$ points of P. We then apply a binary search on the sorted list of these candidate side lengths. This gives us an interval I defined by two consecutive candidate side lengths in the sorted list containing the side length r^* of the largest inner square whose corresponding annulus contains $n-k$ points of P.

We consider the distance between each point $p \in P$ and the line containing ℓ, and take twice the value as a candidate side length. We also consider the difference between $y(p)$ for each $p \in P$ and $y(q)$ for each $q \in \{q_1, q_2\}$, and take twice the value as a candidate side length as well. (Recall that q_1 and q_2 are the two endpoints of ℓ.) Precisely, $2|x(p) - x(\ell)|$, $2|y(p) - y(q_1)|$, and $2|y(p) - y(q_2)|$ are candidate side lengths of the largest inner square for a point $p \in P$. See Figure 2(b) for an illustration. We reject all distances larger than r_0 as no inner square of such

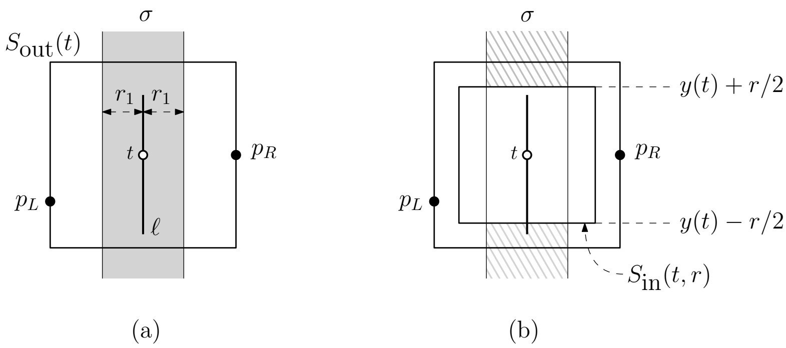

Figure 3: (a) The open region (strip) σ between the two vertical lines at distance r_1 from the line containing ℓ. (b) We have $k_{\text{in}}(t,r) = |P \cap \sigma| - m^+(y(t)+r/2) - m^-(y(t)-r/2)$.

a large side length defines a square annulus. We apply a binary search on the sorted list of all candidate side lengths using the decision algorithm above. As a result, we obtain the interval $I = [r_1, r_2)$ bounded by two consecutive candidate side lengths in the sorted list such that I contains the side length r^* of the largest inner square whose corresponding annulus contains $n - k$ points of P.

4.3.3 Linear search for r^* in I.

Now we have the interval $I = [r_1, r_2)$ containing the side length r^* of the largest inner square. For a side length $r \in I$, consider the square $S_{\text{in}}(t,r)$ of side length r centered at point $t \in \ell$. Imagine that we translate t from one endpoint of ℓ to the other endpoint (and therefore the square $S_{\text{in}}(t,r)$ is translated accordingly.) Then the top side of $S_{\text{in}}(t,r)$ hits points in P. Observe that the set of points in P hit by the top side during the translation remains the same for any value $r \in I$ because there is no point p in P with $r_1 < 2|x(p) - x(\ell)| < r_2$. This implies that the order of the points hit by the top side of $S_{\text{in}}(t,r)$ during the translation remains the same even for varying $r \in I$. This also holds for the bottom side of $S_{\text{in}}(t,r)$.

Let T_{out} (and B_{out}) be the list of points in P hit by the top side (and the bottom side) of $S_{\text{out}}(t)$ sorted by their y-coordinates while t translates along ℓ from its lower endpoint q_2 to its upper endpoint q_1. Also, let T_{in} (and B_{in}) be the list of points in P hit by the top side (and the bottom side) of $S_{\text{in}}(t,r)$ for a fixed value r with $r_1 < r < r_2$ sorted by their y-coordinates while t translate along ℓ from its lower endpoint q_2 to its upper endpoint q_1. As discussed above, T_{in} and B_{in} are uniquely determined, regardless of choice of r. The four lists can be computed in $O(n)$ time, as the points in P are already sorted in their y-coordinates.

For $t \in \ell$, let $k_{\text{out}}(t) := |P \setminus S_{\text{out}}(t)|$ be the number of points in P that lie strictly outside of $S_{\text{out}}(t)$. The function $k_{\text{out}}(t)$ of $t \in \ell$ is piecewise constant, and can be explicitly obtained in $O(n)$ time by performing a linear search using lists T_{out} and B_{out}.

For $t \in \ell$ and $r > 0$, let $k_{\text{in}}(t,r)$ be the number of points in P that lie strictly in the interior of $S_{\text{in}}(t,r)$. Consider two vertical lines that are r_1 distant from the line containing ℓ, and the open region (strip) σ between the two lines. See Figure 3(a).

Observe that every point contained in the interior of $S_{\text{in}}(t,r)$ is also contained in σ because there is no point p in P with $r_1 < 2|x(p) - x(\ell)| < r_2$. Let $m^+(y_1)$ be the number of points in $P \cap \sigma$ whose y-coordinate is at least y_1; $m^-(y_2)$ be the number of points in $P \cap \sigma$ whose y-coordinate is at most y_2. Then, for $r \in I$, we have

$$k_{\text{in}}(t,r) = |P \cap \sigma| - m^+(y(t)+r/2) - m^-(y(t)-r/2).$$

See Figure 3(b) for an illustration. Note that the functions $m^+(y_1)$ and $m^-(y_2)$ are piecewise constant, and can be explicitly specified in $O(n)$ time using lists $T_{\rm in}$ and $B_{\rm in}$.

Our task here is thus to find a maximum $r^* \in I = [r_1, r_2)$ such that there exists some $t \in \ell$ with $k_{\rm out}(t) + k_{\rm in}(t, r^*) \leq k$. For the purpose, we perform a linear search on the precomputed lists $T_{\rm out}$, $B_{\rm out}$, $T_{\rm in}$, and $B_{\rm in}$, according to each of a few possible configurations of an optimal annulus. By Lemma 4, for an optimal annulus $(S_{\rm out}(t), S_{\rm in}(t, r^*))$, either (1) an additional point of P lies on the top or bottom side of the outer square $S_{\rm out}(t)$, or (2) two points of P lie on the top and bottom sides of the inner square $S_{\rm in}(t, r^*)$. We handle each case separately.

First, we describe how to handle case (1). There are two cases: an additional point of P lies on the top side of $S_{\rm out}(t)$ or its bottom side. Assume the former case; the other case can be handled in a symmetric way. By Lemma 4, there is another point of P that lies on a side of the inner square $S_{\rm in}(t, r^*)$. Again, there are two subcases: a point lies (a) on the top side or (b) on the bottom side of the inner square $S_{\rm in}(t, r^*)$.

Assume the former subcase (a). In this case, we maintain two pointers p and p' such that p points to an element in $T_{\rm out}$ and p' points to one in $T_{\rm in}$. These two pointers p and p' initially point to the first ones in $T_{\rm out}$ and $T_{\rm in}$, respectively, and will scan $T_{\rm out}$ and $T_{\rm in}$ in a linear way. As p fixes the top side of $S_{\rm out}(t)$, it determines the center $t \in \ell$ of the annulus, while p' determines the side length r of the inner square $S_{\rm in}(t, r)$ since p' fixes the top side of $S_{\rm in}(t, r)$. In this way, a pair (p, p') determines a square annulus $A(p, p')$. We also maintain the maximum side length of a feasible inner square we have had so far by a variable $r_{\max}$, initially being $r_{\max} = r_1$. For the current pair (p, p') of pointers, we repeat the following until p' reaches the last point in $T_{\rm in}$: If $r \geq r_2$, then we ignore $A(p, p')$ and move p to the next point in $T_{\rm out}$. If $r \leq r_{\max}$, then we ignore $A(p, p')$ and move p' to the next point in $T_{\rm in}$. Otherwise, we have $r_{\max} < r < r_2$, and we test if $k_{\rm out}(t) + k_{\rm in}(t, r) \leq k$. If $k_{\rm out}(t) + k_{\rm in}(t, r) \leq k$, then we set $r_{\max}$ to be r, keep $A(p, p')$ as the best solution we have had so far, and move p' to the next point in $T_{\rm in}$; otherwise, if $k_{\rm out}(t) + k_{\rm in}(t, r) > k$, then we move p to the next point in $T_{\rm out}$.

Each iteration of the above procedure can be implemented in $O(1)$ time as follows: Evaluating $k_{\rm out}(t)$ can be done in $O(1)$ time per iteration since t translates upwards as p scans the points in $T_{\rm out}$. To evaluate $k_{\rm in}(t, r)$ in $O(1)$ time, we also maintain two variables y_1 and y_2 and update them to have $y_1 = y(p')$ and $y_2 = y(t) - r$ at each iteration whenever the evaluation of $k_{\rm in}(t, r)$ is performed. Note that y_1 and y_2 are the y-coordinates of the top and bottom sides of $S_{\rm in}(t, r)$. We observe that y_1 and y_2 are also non-decreasing, and hence that $m^+(y_1)$ and $m^-(y_2)$ can be evaluated in $O(1)$ time at every iteration whenever necessary.

Subcase (b), where a point lies on the bottom side of the inner square, can be handled in an analogous way with two pointers p and p' scanning $T_{\rm out}$ and $B_{\rm in}$, respectively. Consequently, if an optimal annulus falls in case (1), then we can find it in $O(n)$ time.

Next, we consider case (2), in which two points of P lie on the top and bottom sides of the inner square $S_{\rm in}(t, r^*)$. In this case, note that the side length r^* of the inner square is determined by the two points on its top and bottom sides. Similarly to the above case, we maintain two pointers p^+ and p^- scanning two lists $T_{\rm in}$ and $B_{\rm in}$, respectively. Then, the side length r is determined by the difference $y(p^+) - y(p^-)$ of the y-coordinates of p^+ and p^-, and $t \in \ell$ is also determined such that $y(t) = (y(p^+) + y(p^-))/2$. Thus, a pair (p^+, p^-) determines a square annulus as above, denoted by $A(p^+, p^-)$. As done for case (1), we maintain the maximum side length $r_{\max}$ of a feasible inner square we have had so far, initially being $r' = r_1$. For the current pair (p^+, p^-) of pointers, we repeat the following until p^+ reaches the last point in $T_{\rm in}$: If $r \geq r_2$, then we ignore $A(p^+, p^-)$ and move p^- to the next point in $B_{\rm in}$. If $r < r_{\max}$, then we ignore $A(p^+, p^-)$ and move p^+ to the next point in $T_{\rm in}$. Otherwise, we have $r_{\max} \leq r < r_2$, and we test if $k_{\rm out}(t) + k_{\rm in}(t, r) \leq k$. If $k_{\rm out}(t) + k_{\rm in}(t, r) \leq k$, then we set $r_{\max}$ to be r, keep $A(p^+, p^-)$ as the best solution we have had so far, and move p^+ to the next point in $T_{\rm in}$; if $k_{\rm out}(t) + k_{\rm in}(t, r) > k$, then we move p^- to the next point in B_{in}.

Each iteration of the above procedure can be implemented in $O(1)$ time. We observe that $y(t) = (y(p^+) + y(p^-))/2$ is non-decreasing, as $r_{\max}$ is non-decreasing and both p^+ and p^- scan the points in T_{in} and B_{in} linearly. Therefore, $k_{\text{out}}(t)$ can be evaluated in $O(1)$ time per iteration. To evaluate $k_{\text{in}}(t,r)$ in $O(1)$ time, we also maintain two variables y_1 and y_2, and update them to have $y_1 = y(p^+)$ and $y_2 = y(p^-)$ at each iteration whenever the evaluation of $k_{\text{in}}(t,r)$ is performed. We observe that y_1 and y_2 are non-decreasing, so evaluation of $m^+(y_1)$ and $m^-(y_2)$ can be done in $O(1)$ time. Consequently, if an optimal annulus falls in case (2), then we can find it in $O(n)$ time.

We finally conclude the following result.

Theorem 2 *Given a set P of n points in the plane and an integer k with $1 \le k \le n$, a minimum-width square annulus of P with k outliers can be computed in $O(k^2 n \log n)$ time.*

5 Rectangular Annulus with Outliers

In this section, we present two algorithms for computing a minimum-width rectangular annulus of a set P of n points with k outliers for $1 \le k \le n$. Our first algorithm takes $O(n \log n + k^4 \log^2 n)$ time and the second one takes $O(k^2 n \log k + k^4 \log^2 k)$ time.

Our algorithm is based on the following lemma given by Bae [5].

Lemma 6 (Bae [5]) *There exists a minimum-width rectangular annulus of P with k outliers such that each side of its outer rectangle contains a point in P.*

Due to this lemma, we can find $O(k^4)$ candidates of the outer rectangle of an optimal annulus as we did in Section 4.2.

5.1 Finding the smallest-width annulus for a fixed outer rectangle

We assume that we are given a data structure constructed on P that allows us to count the number of points of P lying on a query rectangle in $O(\log n)$ time [8]. Such a data structure can be constructed in $O(n \log n)$ time and has $O(n \log n)$ size. We also assume that we are given two balanced binary search trees constructed on P, one with respect to their x-coordinates and the other with respect to their y-coordinates. Let $\mathcal{T}_x$ denote the balanced binary search tree with respect to the x-coordinates.

Let R be a candidate outer rectangle. Our goal in this subsection is to find the minimum-width annulus whose outer rectangle is R. In other words, we find the inner rectangle with respect to R containing at most $k - k_{\text{out}}$ points of P, where k_{out} is the number of points in P lying outside of R. Recall that a rectangular annulus is determined by its outer and inner rectangles and the inner rectangle is an inward offset of the outer rectangle.

Given a value $\delta \ge 0$, we can determine in $O(\log n)$ time whether the annulus with outer rectangle R of width at most δ contains at least $n - k$ points by checking whether the inward offset of R by δ contains less than or equal to $k - k_{\text{out}}$ points of P using the data structure for counting queries. This is our decision algorithm for a fixed width $\delta \ge 0$.

Let δ^* be the minimum width such that our decision algorithm returns a positive answer, and R^* be the inward offset of R by δ^*. That is, the annulus determined by R and R^* is the optimal solution for fixed outer rectangle R, and its width is δ^*. To reduce the search space for δ^*, we make use of the observation that at least one side of R^* contains a point of P. Consider the case that the left side of R^* contains a point $p^* \in P$. Let x_1 be the x-coordinate of the left side of R and x_2 be the x-coordinate of the center of R. Then, it is obvious that the x-coordinate of p^* lies in the interval $[x_1, x_2]$.

Now, we are ready to describe our algorithm to find p^* and δ^*. It starts with two standard queries for x_1 and x_2 on the balanced binary search tree $\mathcal{T}_x$ on P with respect to the x-coordinates, resulting in two paths from the root to a leaf in $\mathcal{T}_x$. The two search paths share a common part from the root and then split at some node v of $\mathcal{T}_x$. We traverse $\mathcal{T}_x$ again from the split node v. By the construction, the x-coordinate x_v corresponding to v lies in $[x_1, x_2]$. We then apply our decision algorithm for $\delta = x_v - x_1$. If the result is positive, we proceed to the left child of v; otherwise, we proceed to the right child of v. We apply our decision algorithm for the next node repeatedly until we reach a leaf of $\mathcal{T}_x$. Then, the leaf node corresponds to the point p^*, in this case. Since the height of $\mathcal{T}_x$ is $O(\log n)$ and our decision algorithm takes $O(\log n)$ time, this procedure terminates in $O(\log^2 n)$ time.

The other cases, where p^* lies on the right, top, or bottom side of R^*, can be handled in a symmetric way by traversing the binary search tree on P with respect to the x-coordinates or y-coordinates.

The following summarizes the above discussion.

Lemma 7 *Given a fixed outer rectangle R, we can compute in $O(\log^2 n)$ time the smallest width δ^* such that R and its inward offset by δ^* form a rectangular annulus with k outliers after $O(n \log n)$ time preprocessing.*

5.2 Putting it all together

Since we have $O(k^4)$ candidate outer rectangles by Lemma 6, we obtain the following by the above discussion.

Theorem 3 *Given a set P of n points in the plane and an integer k with $1 \leq k \leq n$, a minimum-width rectangular annulus of P with k outliers can be computed in $O(n \log n + k^4 \log^2 n)$ time.*

Proof. As a preprocessing, we build a data structure on P that allows orthogonal range counting queries in $O(\log n)$ time, and build two balanced binary search trees on P with respect to the x-coordinates and the y-coordinates, respectively. This can be done in $O(n \log n)$ time by standard data structures [8].

Then we gather the set of $O(k^4)$ candidate outer rectangles by Lemma 6. For each candidate outer rectangle, we apply our algorithm described in Section 5.1 to compute the smallest possible width in $O(\log^2 n)$ time. By choosing the smallest one among the computed widths, we find a minimum-width rectangular annulus of P with k outliers. The total time complexity is $O(n \log n + k^4 \log^2 n)$. ◻

The time bound in Theorem 3 has a term of $n \log n$, and this does not match the case of $k = 0$ in which one can solve the problem in $O(n)$ time [1]. In order to reduce the running time for small k, we exploit the approach by Bae [5].

A subset $K \subseteq P$ is called a *kernel* for P if a minimum-width rectangular annulus of K with k outliers is also a minimum-width rectangular annulus of P with k outliers. Bae [5] presented a procedure to compute a kernel K of size $O(k^4)$ in $O(nk^2 \log k + k^4)$ time. After computing such a kernel K, we compute a minimum-width rectangular annulus of K with k outliers in $O(k^4 \log^2 k)$ time using Theorem 3. Hence, we conclude the following theorem.

Theorem 4 *Given a set P of n points in the plane and an integer k with $1 \leq k \leq n$, a minimum-width rectangular annulus of P with k outliers can be computed in $O(nk^2 \log k + k^4 \log^2 k)$ time.*

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