Maximizing the Overlap of Two Planar Convex Sets under Rigid Motions ⋆

Hee-Kap Ahn [a], Otfried Cheong [b,∗], Chong-Dae Park [b]
Chan-Su Shin [c], Antoine Vigneron [d]

[a] *Department of Computer Science and Engineering, Sejong University, Seoul, Korea.*

[b] *Division of Computer Science, Korea Advanced Institute of Science and Technology, Daejeon, Korea.*

[c] *School of Electr. and Inform. Engineering, Hankuk University of Foreign Studies, Yongin, Korea.*

[d] *Unité Mathématiques et Informatique Appliquées, INRA, Jouy-en-Josas, France.*

Abstract

Given two compact convex sets P and Q in the plane, we compute an image of P under a rigid motion that approximately maximizes the overlap with Q. More precisely, for any $\varepsilon > 0$, we compute a rigid motion such that the area of overlap is at least $1-\varepsilon$ times the maximum possible overlap. Our algorithm uses $O(1/\varepsilon)$ extreme point and line intersection queries on P and Q, plus $O((1/\varepsilon^2)\log(1/\varepsilon))$ running time. If only translations are allowed, the extra running time reduces to $O((1/\varepsilon)\log(1/\varepsilon))$. If P and Q are convex polygons with n vertices in total that are given in an array or balanced tree, the total running time is $O((1/\varepsilon)\log n + (1/\varepsilon^2)\log(1/\varepsilon))$ for rigid motions and $O((1/\varepsilon)\log n + (1/\varepsilon)\log(1/\varepsilon))$ for translations.

Key words: Approximation algorithm, sublinear algorithm, convex shape, geometric pattern matching

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∗ Corresponding author.

Email addresses: `heekap@gmail.com` (Hee-Kap Ahn), `otfried@kaist.ac.kr` (Otfried Cheong), `cdpark@jupiter.kaist.ac.kr` (Chong-Dae Park), `cssin@hufs.ac.kr` (Chan-Su Shin), `antoine.vigneron@jouy.inra.fr` (Antoine Vigneron).

1 Introduction

We consider the following problem: given two compact convex sets P and Q in the plane, find a rigid motion φ such that the area of $\varphi P \cap Q$ is maximized, where φP is the image of P under φ.

The area of overlap (or, equivalently, the area of the symmetric difference) of two planar regions is a natural measure of their similarity that is insensitive to noise [4,6]. Most previous theoretical work on the problem has restricted φ to be a translation. De Berg et al. [6] gave an $O(n \log n)$ time algorithm to solve the problem for two convex polygons with n vertices in total, making use of the Brunn-Minkowski theorem, and gave a constant-factor approximation. If φ is restricted to be a translation along a given line, then a linear time algorithm is possible [3]. Alt et al. [4] gave a constant-factor approximation for the minimum area of the symmetric difference.

More general objects have also been considered. Mount et al. [10] studied the function mapping a translation vector to the area of overlap of a translated simple n-vertex polygon P with another simple m-vertex polygon Q, showing that it is continuous, piecewise polynomial of degree at most two, has $O((nm)^2)$ pieces, and can be computed within the same time bound. No algorithm is known that computes the translation maximizing the area of overlap that does not essentially construct the whole function graph. De Berg et al. [5] consider the case where P and Q are disjoint unions of n and m unit disks, with $n \leqslant m$. They compute a $(1-\varepsilon)$-approximation for the maximal area of overlap of P and Q under translations in time $O((nm/\varepsilon^2)\log(m/\varepsilon))$.

In contrast, surprisingly little is known about the problem if φ can be any rigid motion. Alt et al. [2] made some initial progress on a similar problem, showing, for instance, how to construct, for a convex polygon P, the axis-parallel rectangle Q minimizing the symmetric difference of P and Q. In the case where P and Q are disjoint unions of n and m unit disks, de Berg et al. [5] compute a $(1-\varepsilon)$-approximation for the maximal area of overlap of P and Q under rigid motions in time $O((n^2m^2/\varepsilon^3)\log m)$. Dickerson and Scharstein [8] consider the case where P is a convex m-gon, and Q a set of n points in the plane, and show how to find a rigid motion of P that contains the maximum number of points in Q. Finally, Cheong et al. [7] gave a general framework for maximizing the overlap of two shapes. This framework can be applied to convex polygons, but computes an approximation with *absolute* error, that is, a rigid motion φ such that the area of $\varphi P \cap Q$ is at least the optimal area minus ε times the area of P. No algorithm is known that solves the problem *exactly*. The standard approach of decomposing the configuration space into regions where the intersection of the two polygons is combinatorially invariant does not easily lead to such an algorithm. The difficulty is that the function expressing the area of intersection for a known combinatorial type is complicated, and it is not clear whether the maximum of this function can be found exactly under a realistic model of computation.

Ahn et al. [1] recently gave an algorithm to find an approximation to the largest axially-symmetric convex polygon included in a given convex polygon P. This can be considered a special case of our problem, where Q is a reflected copy of P. Their solution exploits this special relationship between Q and P, and does not generalize to our more general problem. Indeed, the hardest case in the analysis of our algorithm is when P is a rather "round" polygon, and Q is long and skinny—this cannot happen in their setting.

We give an algorithm that, given any $\varepsilon > 0$, computes a rigid motion φ such that the area of $\varphi P \cap Q$ is at least $1 - \varepsilon$ times the maximum possible area. It performs $O(1/\varepsilon)$ extreme point and line intersection queries on the convex sets P and Q, and requires additional time $O((1/\varepsilon^2)\log(1/\varepsilon))$.

Our algorithm is in fact surprisingly simple. Given two polygons P and Q with n vertices in total, we generate a set of $O(1/\varepsilon)$ orientations for P, and run the $O(n \log n)$ algorithm of de Berg et al. [6] to compute the optimal translation for each orientation. The total running time of this procedure is $O((n \log n)/\varepsilon)$, and the difficulty lies entirely in the selection of appropriate orientations—uniform sampling does not work, and the set of orientations needs in fact to be chosen based on the aspect ratios of P and Q—and in proving the approximation bound.

Like Ahn et al. [1], we then show that we can replace the convex input sets P and Q by polygonal inner approximations, whose complexity depends on ε only (and not on the complexity of the input sets). This simplification can be done entirely using two kinds of queries on the input sets, namely intersecting queries with a line, and finding the point extreme in a given direction. In the problems studied by Ahn et al. [1], the well-known Dudley approximation with $O(1/\sqrt{\varepsilon})$ vertices could be used as this inner approximation. In this paper, we introduce two new approximations that are stronger than Dudley's in two different senses, but unfortunately both require $\Theta(1/\varepsilon)$ vertices. The same inner approximations can be used to approximately solve the problem of maximizing the overlap of planar convex sets under *translations*. This requires $O(1/\varepsilon)$ extreme point and line intersection queries on the convex sets P and Q, and additional time $O((1/\varepsilon)\log(1/\varepsilon))$.

If P and Q are convex n-gons, given as an array or balanced tree containing the vertices in sorted order, then the two queries can be implemented in $O(\log n)$ time. The running time of our algorithm is then $O((1/\varepsilon)\log n + (1/\varepsilon^2)\log(1/\varepsilon))$ for rigid motions and $O((1/\varepsilon)\log n + (1/\varepsilon)\log(1/\varepsilon))$ for translations. If the convex n-gons are given differently, for instance as a linked list, then an additional

$O(n)$ term has to be added.

The reader may wonder why we cannot simply use the "naive" approach of first approximating both sets using Dudley's approximation, and then running an exact algorithm on the approximations. This approach fails for two reasons: First, as mentioned above, no exact algorithm is known, and if one is found, it is likely to be quite complicated and impractical. Second, we have not been able to prove that an exact solution to the problem for Dudley's approximation guarantees a $(1-\varepsilon)$-approximation for the original sets (the proof of Claim (1) in Section 5 breaks down if we use Dudley's approximation). It would be nice to either prove this (as it would also improve the time bound of our algorithm by a factor of $O(1/\sqrt{\varepsilon})$), or to find a counter-example.

2 Preliminaries

Let C denote a compact convex set in the plane. We let $|C|$ and $d(C)$ denote the area and diameter of C. Let $w(C)$ denote the width of C, that is, the minimum distance between two parallel lines enclosing C. We call a pair of points (p, q) in C an *antipodal grasp* of C if C lies inbetween the lines through p and q orthogonal to pq, see Figure 1. The width is always achieved by an

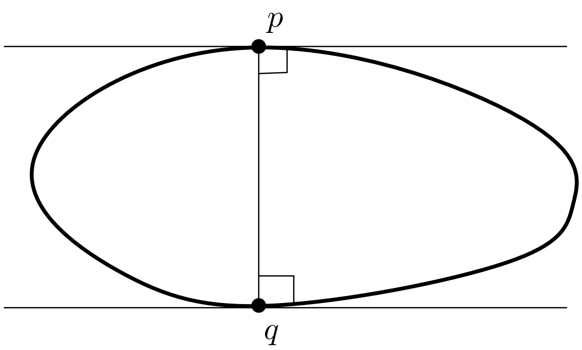

Fig. 1. An antipodal grasp.

antipodal grasp, as the following lemma shows.

Lemma 1 *Let C be a compact convex set in the plane. There is an antipodal grasp (p, q) of C such that the segment $pq \subset C$ has length $w(C)$. If C is a convex n-gon, then such an antipodal grasp can be found in $O(n)$ time.*

PROOF. Let ℓ_1 and ℓ_2 be two parallel lines that achieve the width. Let $s_1 = \ell_1 \cap C$ and $s_2 = \ell_2 \cap C$ (by convexity of C, these sets are either points or segments). Assume that there is no pair of points $(p, q) \in s_1 \times s_2$ such that pq is orthogonal to ℓ_1 and ℓ_2. (See Figure 2.) Since s_1 and s_2 are convex, this implies that there is a line ℓ orthogonal to ℓ_1 and ℓ_2 such that s_1 and

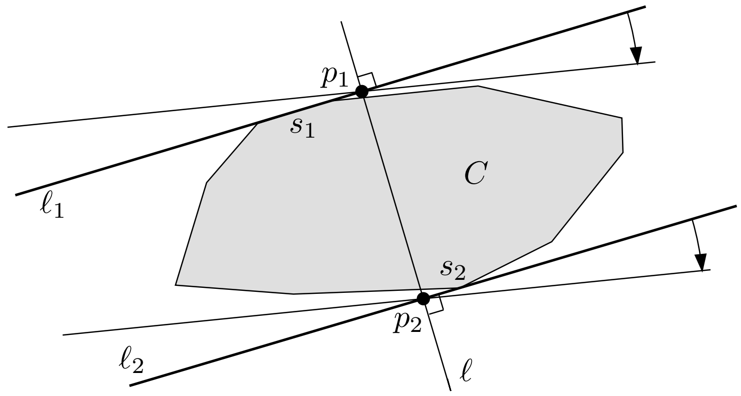

Fig. 2. Proof of Lemma 1.

s_2 lie strictly on different sides of ℓ. Let p_1 (resp. p_2) denote the intersection of ℓ with ℓ_1 (resp. ℓ_2). If we rotate ℓ_1 slightly around p_1 towards s_2, and ℓ_2 around p_2 towards s_1, we obtain two parallel lines enclosing C whose distance is less than $w(C)$, a contradiction.

It follows that an antipodal grasp (p, q) realizing the width exists. In case C is a convex n-gon, we can find it in $O(n)$ time using the rotating calipers technique of Toussaint [12]. $\square$

We will call a segment pq as in Lemma 1 a *spine* of C. A similar statement for the diameter is well-known:

Lemma 2 *Let C be a compact convex set in the plane. Any segment $pq \subset C$ of length $d(C)$ defines an antipodal grasp (p, q).*

Width and diameter allow us to approximate the area of a convex set.

Lemma 3 *Let C be a compact convex set in the plane. Then $w(C)d(C)/2 \leqslant |C| \leqslant w(C)d(C)$.*

PROOF. Let R be the rectangle circumscribed to C with two sides parallel to a spine of C and such that C touches all four sides of R. The sides of R have length $w(C)$ and $d \leqslant d(C)$, and so $|C| \leqslant |R| = w(C)d \leqslant w(C)d(C)$.

Let now R' be the rectangle circumscribed to C with two sides parallel to a diameter of C and such that again C touches all four sides of R', see Figure 3. The sides of R' have length $d(C)$ and $w \geqslant w(C)$. C contains two triangles with a common base of length $d(C)$, and total height w. This implies $|C| \geqslant d(C)w/2 \geqslant w(C)d(C)/2$. $\square$

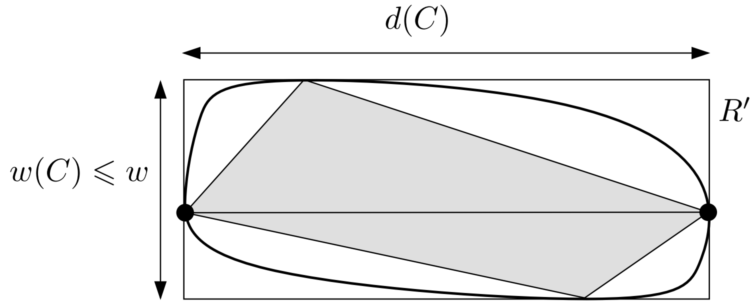

Fig. 3. Proof of Lemma 3.

Diameter, width, and area of a convex n-gon can all be computed in linear time [12], and this is optimal. To achieve sublinear algorithms, and to handle non-polygonal convex sets, we need to be able to process two queries on convex sets. Given a convex set C, let T_C be the time to answer the following two kinds of queries: (a) given a direction vector u, find the point v in C extreme in direction u, that is, maximizing the dot product $\langle v, u\rangle$; and (b) given a line ℓ, find the intersection $\ell \cap C$ (a line segment). Obviously, T_C depends on how C is represented—for instance, $T_C = O(\log n)$ if C is a convex n-gon stored in an array (in sorted order), but $T_C = \Theta(n)$ if it is stored in a linked list.

The following lemma allows us to compute rough estimates of diameter, width, and area of a convex set using these queries. Let $\mathrm{dist}(p, q)$ denote the Euclidean distance between points p and q.

Lemma 4 *Let C be a compact convex set in the plane. In $O(T_C)$ time we can find two rectangles r and R such that:*

(i) *$r \subset C \subset R$, with C touching all four sides of R.*
(ii) *r and R are homothetic, with a homothety ratio $3\sqrt{2}$.*
(iii) *Let d and w be the lengths of the sides of R, with $d \geqslant w$. Then $d(C)/\sqrt{2} \leqslant d \leqslant d(C)$, $w(C) \leqslant w \leqslant 2\sqrt{2}w(C)$, and $|R|/(2\sqrt{2}) \leqslant |C| \leqslant |R|$.*

PROOF. By doing four queries, we can find the axis-parallel bounding box R' of C. Let $a' \geqslant b' > 0$ be its sides, and pick the vertices p, q of C touching the sides of R' of length b', see Figure 4. Then $a' \leqslant \mathrm{dist}(p, q) \leqslant d(C) \leqslant d(R) \leqslant \sqrt{2}a'$. Using four more queries, we now find the smallest rectangle R containing C with two sides parallel to pq. These sides have length $a \geqslant \mathrm{dist}(p, q) \geqslant a' \geqslant d(C)/\sqrt{2}$, and the other sides have length b. Since C contains two triangles with a common base pq and total height b, we have $ab = |R| \geqslant |C| \geqslant \mathrm{dist}(p, q)b/2 \geqslant a'b/2 \geqslant (d(C)/\sqrt{2})b/2 \geqslant ab/(2\sqrt{2}) = |R|/(2\sqrt{2})$. From $d(C)/\sqrt{2} \leqslant a' \leqslant \mathrm{dist}(p, q)$ follows $d(C)b \leqslant \sqrt{2}\mathrm{dist}(p, q)b \leqslant 2\sqrt{2}|C| \leqslant 2\sqrt{2}w(C)d(C)$, so $b \leqslant 2\sqrt{2}w(C)$.

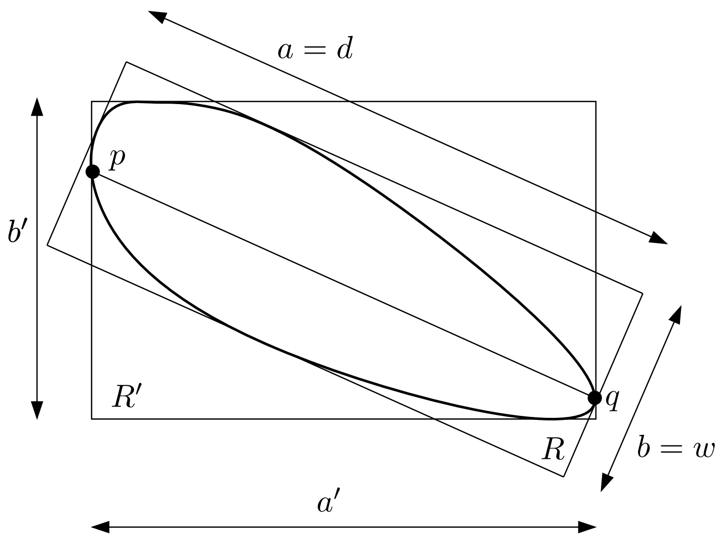

Fig. 4. Proof of Lemma 4.

Let now $d := \max(a, b)$, $w := \min(a, b)$. We have $d \leqslant d(C)$, since there are points of C on each pair of opposite sides of R, and $w \geqslant w(C)$, since C is contained inbetween two parallel lines at distance w. So $d \geqslant a \geqslant d(C)/\sqrt{2}$, and $w \leqslant b \leqslant 2\sqrt{2}w(C)$, which completes the proof of (iii).

Now we show how to find r. Recall that $\mathrm{dist}(p, q) \geqslant d(C)/\sqrt{2} \geqslant a/\sqrt{2}$, and C contains two triangles with a common base pq and total height b. Note that one of these has height at least $b/2$. If it has an obtuse interior angle around p (resp. q), then it contains a homothet r of R with homothety ratio at least $1/(3\sqrt{2})$ such that r has a corner at p (resp. q). Otherwise, it contains a homothet r of R with homothety ratio at least $1/4$ such that r has one side lying in pq. Since we know one point of C touching each side of R, we can find r in constant time. □

Note that with more effort, better estimates can be obtained. For instance, with $O(1/\sqrt{\varepsilon})$ queries, a $(1 + \varepsilon)$-approximation to diameter, width, and area can be computed [1], and a homothety ratio of $2 + \varepsilon$ can be obtained [11].

The following lemma gives a somewhat larger inscribed rectangle, but does not permit a sublinear construction. (Alternatively, one could show the existence of an inscribed rectangle with sides $d(C)/(2\sqrt{2})$ and $w(C)/2$ using the results by Schwarzkopf et al. [11]).

Lemma 5 *Let C be a convex set in the plane. There is a rectangle r with sides $d(C)/2$ and $w(C)/4$ contained in C. If C is a convex n-gon, then we can compute r in $O(n)$ time.*

PROOF. Let pq be a diameter of C, and let R be a rectangle circumscribed to C with two sides parallel to pq such that C touches all four sides of R. Let w be the side of R orthogonal to pq, see Figure 5. Then C contains two triangles

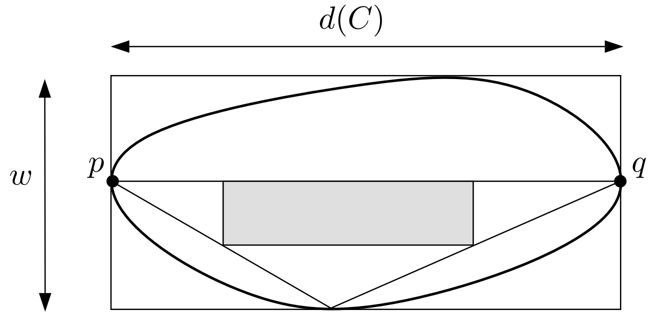

Fig. 5. Proof of Lemma 5.

with a common base pq and total height w. One of these has height at least $w/2$, and contains a rectangle of length $\mathrm{dist}(p,q)/2 = d(C)/2$ and height at least $w/4 \geqslant w(C)/4$.

If C is a convex n-gon, then we can compute its diameter in $O(n)$ time [12], and it is easy to find r in $O(n)$ time. □

This implies the following lower bound on the overlap of two convex sets under rigid motions.

Lemma 6 *Let C_1 and C_2 be convex sets in the plane. There is a rigid motion φ such that*

$$|\varphi C_1 \cap C_2| \geqslant \frac{1}{8} \cdot \min\{d(C_1), d(C_2)\} \cdot \min\{w(C_1), w(C_2)\}.$$

PROOF. By Lemma 5 there are rectangles $r_i \subset C_i$ of size $d(C_i)/2 \times w(C_i)/4$, for $i = 1, 2$. Let φ be the rigid motion maximizing $|\varphi r_1 \cap r_2|$. Then

$$|\varphi r_1 \cap r_2| \geqslant \min\{d(C_1)/2, d(C_2)/2\} \cdot \min\{w(C_1)/4, w(C_2)/4\},$$

and the lemma follows. □

If two convex sets are long and skinny, the rigid motion achieving maximal overlap must align them rather well.

Lemma 7 *Let C_1 and C_2 be convex sets in the plane, let φ^{opt} be the rigid motion maximizing $|\varphi^{\mathrm{opt}} C_1 \cap C_2|$, and let ϑ be the angle between spines of*

$\varphi^{\mathrm{opt}} C_1$ *and* C_2. *Then*

$$\sin\vartheta \leqslant \frac{8\max\{w(C_1), w(C_2)\}}{\min\{d(C_1), d(C_2)\}}.$$

PROOF. The set C_i is contained in an infinite strip of width $w(C_i)$, for $i = 1, 2$. When these two strips make an angle of $\vartheta > 0$, their intersection is a parallelogram of area $w(C_1)w(C_2)/\sin\vartheta$. By Lemma 6, this area must be at least $\min\{w(C_1), w(C_2)\}\min\{d(C_1), d(C_2)\}/8$. This implies the lemma. □

We will need two more results. The first one is from Ahn et al. [1], the second one is by de Berg et al. [6] and makes use of the Brunn-Minkowski theorem.

Lemma 8 ([1]) *Let C be a convex set, and let C' be a copy of C, rotated by an angle δ around a point p in C. Then*

$$|C \cap C'| \geqslant |C| - \frac{\pi\delta}{2} d(C)^2,$$

or, equivalently,

$$|C \setminus C'| \leqslant \frac{\pi\delta}{2} d(C)^2.$$

Lemma 9 ([6]) *Given convex polygons P and Q in the plane with n vertices in total, one can find in time $O(n \log n)$ the translation τ maximizing $|\tau P \cap Q|$.*

3 An algorithm for convex polygons

Let P and Q be convex polygons with n vertices in total, and let $\varepsilon > 0$. Our goal is to find a rigid motion φ^{app} such that $|\varphi^{\mathrm{app}} P \cap Q| \geqslant (1-\varepsilon)|\varphi^{\mathrm{opt}} P \cap Q|$, where φ^{opt} is the rigid motion maximizing $|\varphi P \cap Q|$.

We do this by computing a set of $O(1/\varepsilon)$ orientations for P, and the optimal translation for each orientation using Lemma 9. To show the correctness of this approach, we need to prove that starting with the optimal placement of P (that is, with $\varphi^{\mathrm{opt}} P$), we can rotate P either way by a certain amount and lose only $\varepsilon|\varphi^{\mathrm{opt}} P \cap Q|$. We distribute the proof over the following two key lemmas.

Lemma 10 *Let C_1, C_2 be convex sets with $w(C_2) \leqslant w(C_1)$, let $\varepsilon > 0$, and let*

$$\delta \leqslant \frac{\varepsilon}{4\pi} \frac{w(C_2)}{\min\{d(C_1), d(C_2)\}}.$$

Then there are clockwise and counter-clockwise rotations ρ with angle δ such that $|\rho\varphi^{\rm opt}C_1 \cap C_2| \geqslant (1-\varepsilon)|\varphi^{\rm opt}C_1 \cap C_2|$.

PROOF. In fact, any rotation ρ of angle δ around a point in $\varphi^{\rm opt}C_1 \cap C_2$ will do. To simplify the presentation, we assume that C_1 is already in the optimal placement, that is that $\varphi^{\rm opt}$ is the identity.

If $d(C_1) \leqslant d(C_2)$ then it suffices to show that $|(C_1 \cap C_2) \setminus (\rho C_1 \cap C_2)| \leqslant \varepsilon w(C_2)d(C_1)/8$ by Lemma 6. We observe that $(C_1 \cap C_2) \setminus (\rho C_1 \cap C_2) \subset C_1 \setminus \rho C_1$. By Lemma 8, $|C_1 \setminus \rho C_1| \leqslant \frac{\pi\delta}{2} d(C_1)^2 \leqslant \frac{\pi\varepsilon}{8\pi}\frac{w(C_2)}{d(C_1)} d(C_1)^2 = \frac{\varepsilon}{8} w(C_2)d(C_1)$, as required.

If $d(C_1) > d(C_2)$, we first observe that $|\rho C_1 \cap C_2| = |C_1 \cap \rho^{-1}C_2|$. By Lemma 6, it suffices to show that $|(C_1 \cap C_2) \setminus (C_1 \cap \rho^{-1}C_2)| \leqslant \varepsilon w(C_2)d(C_2)/8$. We have $(C_1 \cap C_2) \setminus (C_1 \cap \rho^{-1}C_2) \subset C_2 \setminus \rho^{-1}C_2$, and by Lemma 8 $|C_2 \setminus \rho^{-1}C_2| \leqslant \frac{\pi\delta}{2} d(C_2)^2 \leqslant \frac{\pi\varepsilon}{8\pi}\frac{w(C_2)}{d(C_2)} d(C_2)^2 = \frac{\varepsilon}{8} w(C_2)d(C_2)$. $\square$

Lemma 11 *Let C_1, C_2 be convex sets with $w(C_2) \leqslant w(C_1)/4$ and $d(C_2) \geqslant d(C_1)/2$, let $\varepsilon > 0$, and let $\delta \leqslant \varepsilon \frac{1}{160}\frac{w(C_1)}{d(C_1)}$. Then there are clockwise and counter-clockwise rotations ρ with angle δ such that $|\rho\varphi^{\rm opt}C_1 \cap C_2| \geqslant (1-\varepsilon)|\varphi^{\rm opt}C_1 \cap C_2|$.*

PROOF. Again, let us assume that C_1 is already in the optimal placement, that is that $\varphi^{\rm opt}$ is the identity. It suffices to show the existence of the clockwise rotation—the existence of the counter-clockwise rotation then follows by applying the lemma to mirror images of C_1 and C_2.

Let S be an infinite strip of width $w(C_2)$ containing C_2, and choose a coordinate system such that S is horizontal. Let R be the smallest axis-parallel bounding rectangle for C_1. We can assume that the distance between the upper edges of S and R is larger than the distance between the lower edges (otherwise we rotate the coordinate system by 180°). Since $w(C_2) \leqslant w(C_1)/4$, and the height of R is at least $w(C_1)$, this implies that the distance between the upper edges is at least $\frac{3}{8}w(C_1)$. Let R' be that part of R that has distance at least $w(C_1)/4$ from the upper edge of R, see Figure 6. The distance between the upper edges of R' and S is still at least $\frac{1}{8}w(C_1)$. The horizontal width of R is at most $d(C_1)$. This implies that a line with absolute slope less than $\frac{1}{4}\frac{w(C_1)}{d(C_1)}$ cannot intersect both the upper edge of R and R'. It follows that a line that is tangent to C_1 from above in a point $u \in R'$ has absolute slope at least $\frac{1}{4}\frac{w(C_1)}{d(C_1)}$.

Let now ρ be the rotation by angle δ, in clockwise direction, around the intersection p of the lower edge of S with the left edge of R. Since $d(C_2) \geqslant d(C_1)/2$,

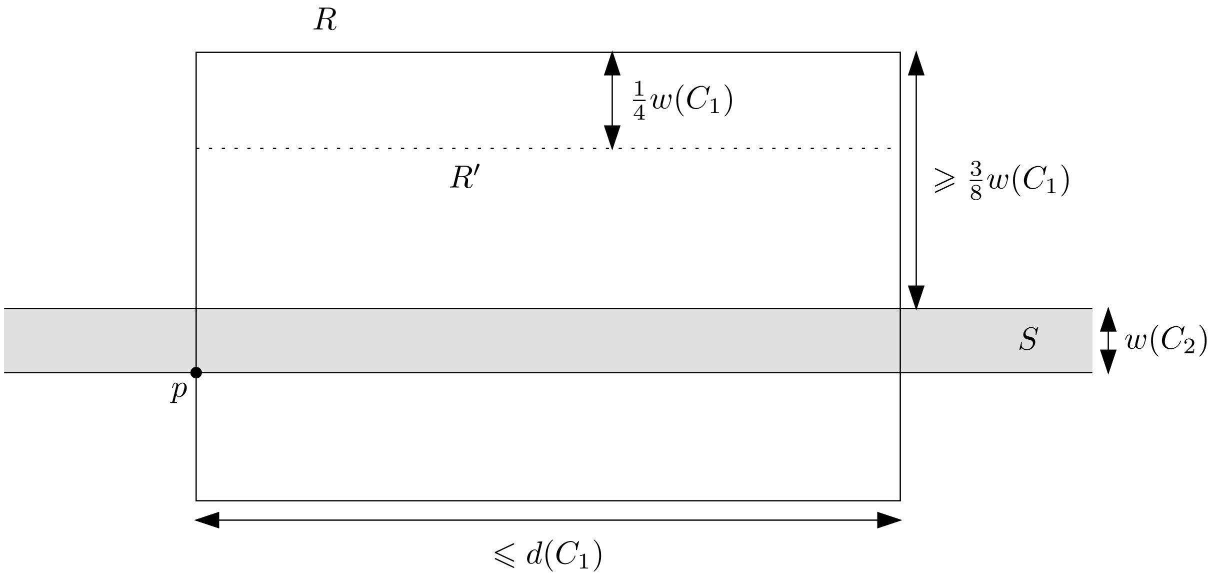

Fig. 6. The strip S and the rectangle R.

it suffices by Lemma 6 to show that $|(C_1 \cap C_2) \setminus (\rho C_1 \cap C_2)| \leqslant \varepsilon w(C_2) d(C_1)/16$.

Let $X := (C_1 \cap C_2) \setminus (\rho C_1 \cap C_2) = (C_1 \setminus \rho C_1) \cap C_2$, and consider a point $q \in X$. Clearly $q \in C_1$. We will show below that the horizontal distance between q and the boundary of C_1 is at most $\varepsilon d(C_1)/32$. Since C_1 is convex, a horizontal line ℓ intersects it in an interval (if at all). The points of $X \cap \ell$ lie in the leftmost and rightmost piece of this interval, in two subintervals of total length at most $\varepsilon d(C_1)/16$. Since $q \in C_2$, we have $q \in S$, and so it suffices to integrate over all horizontal lines in S to establish $|X| \leqslant \varepsilon w(C_2) d(C_1)/16$, as desired.

It remains to prove the following *claim*: the horizontal distance between a point $q \in X$ and the boundary of C_1 is at most $\varepsilon d(C_1)/32$. We observe that $q \in X$ implies $q \notin \rho C_1$, and therefore $\rho^{-1} q \notin C_1$. Let $q' := \rho^{-1} q$. The claim is true if q lies within horizontal distance $\varepsilon d(C_1)/32$ from the left edge of R, so let us assume that is not the case. This implies that the angle that the line pq makes with a vertical line is at least

$$\begin{aligned} \arctan\left(\frac{\varepsilon d(C_1)}{32 w(C_2)}\right) &\geqslant \arctan\left(\frac{\varepsilon w(C_1)}{32} \frac{4}{w(C_1)}\right) \\ &\geqslant \arctan\left(\frac{\varepsilon}{8}\right) \geqslant \frac{\varepsilon}{160} \geqslant \delta, \end{aligned}$$

thus q' lies above and to the left of q (that is, has smaller x-coordinate but larger y-coordinate). Since $q \in C_1$, but $q' \notin C_1$, the segment qq' must intersect the boundary of C_1. The segment has length qq' at most $\delta d(S \cap R) \leqslant \delta(d(C_1) + w(C_2)) \leqslant \frac{5}{4}\delta d(C_1) \leqslant \frac{1}{128}\varepsilon w(C_1)$. If it intersects the *lower* boundary of C_1, we are done: the boundary must pass above q' but below q, and therefore it must intersect the horizontal line through q to the left of q, at a distance smaller than the distance between q and q', which is less than $\varepsilon d(C_1)/32$.

This leaves the case where qq' intersects the upper boundary of C_1. Let u be the point of intersection, and let ℓ be a tangent to C_1 in u. Since the length of qq' is less than $\frac{1}{8}w(C_1)$ and $q \in S$, we have $u \in R'$. As we observed before, this implies that ℓ has absolute slope at least $\frac{1}{4}\frac{w(C_1)}{d(C_1)}$. This implies that ℓ intersects the horizontal line through q in a point u' at distance at most $\varepsilon d(C_1)/32$ from q, see Figure 7. Since C_1 lies below the line ℓ, the boundary of C_1 must intersect the horizontal line through q between q and u', implying the claim, and therefore the lemma. $\square$

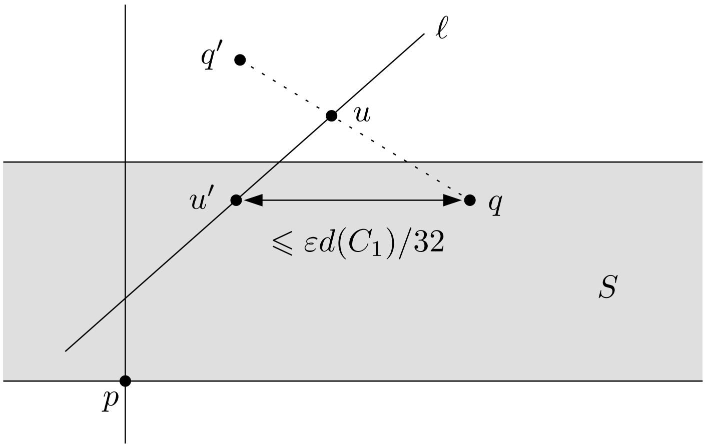

Fig. 7. $\mathrm{dist}(u', q) \leqslant \varepsilon d(C_1)/32$.

We can now describe the algorithm in detail. We start by computing the diameter, width, and a spine of both polygons, in total time $O(n)$. Without loss of generality, let $w(Q) \leqslant w(P)$. If $w(Q) \leqslant w(P)/4$ and $d(Q) \leqslant d(P)/2$, then using Lemma 5 we compute in $O(n)$ time a rectangle $r_P \subset P$ with edge lengths $w(Q)$ and $d(Q)$. By Lemma 1 we can also find in $O(n)$ time a rectangle R_Q that is circumscribed to Q and has edge lengths $w(Q)$ and $d_Q \leqslant d(Q)$. In constant time, we can find a rigid motion φ^{opt} such that $\varphi^{\mathrm{opt}} R_Q \subset r_P$, and hence $\varphi^{\mathrm{opt}} Q \subset P$, which is optimal.

If this is not the case, we sample orientations of P at an interval of $\Delta\varepsilon$, where

$$\Delta := \frac{1}{160} \frac{w(P)}{\min\{d(P), d(Q)\}},$$

but omitting all orientations where the angle ϑ of the spines of P and Q is such that $\sin\vartheta > 1280\Delta$. This results in a set of $O(1/\varepsilon)$ orientations of P. For each of these, we compute the optimal translation using Lemma 9, and retain the best rigid motion found as φ^{app}.

The running time of this procedure is $O((n \log n)/\varepsilon)$, and it remains to prove the approximation bound. By Lemma 7, the spines of $\varphi^{\mathrm{opt}} P$ and Q make an

angle ϑ with
$$\sin\vartheta \leqslant 8\frac{w(P)}{\min\{d(P),d(Q)\}} = 1280\Delta,$$
and so we know that we are sampling an orientation of P at an angle $\delta \leqslant \Delta\varepsilon/2$ from the orientation of $\varphi^{\mathrm{opt}}P$.

If $w(Q) > w(P)/4$, then
$$\Delta = \frac{1}{160}\frac{w(P)}{\min\{d(P),d(Q)\}} < \frac{1}{40}\frac{w(Q)}{\min\{d(P),d(Q)\}} < \frac{1}{4\pi}\frac{w(Q)}{\min\{d(P),d(Q)\}},$$
and so $\delta \leqslant \Delta\varepsilon$ fulfills the assumption of Lemma 10, implying the approximation bound.

If $w(Q) \leqslant w(P)/4$, then we have already excluded the case $d(Q) \leqslant d(P)/2$. We therefore have $\min\{d(P), d(Q)\} > d(P)/2$, which implies
$$\Delta = \frac{1}{160}\frac{w(P)}{\min\{d(P),d(Q)\}} < \frac{2}{160}\frac{w(P)}{d(P)}.$$
Since $\delta \leqslant \Delta\varepsilon/2$, the assumptions of Lemma 11 are fulfilled, and the approximation bound follows.

Lemma 12 *Given two convex polygons P and Q with n vertices in total, and an $\varepsilon > 0$, we can compute a rigid motion φ^{app} such that $|\varphi^{\mathrm{app}}P \cap Q| \geqslant (1-\varepsilon)\max_{\varphi}|\varphi P \cap Q|$, where the maximum is taken over all rigid motions. The running time is $O((n\log n)/\varepsilon)$.*

4 Inner approximations of convex sets

To drastically improve the running time of the algorithm of the previous section, and to apply it to non-polygonal convex sets, we will replace the given convex sets by polygonal approximations whose size depends only on ε. For two sets A and B such that $A \subset B$, the *Hausdorff-distance* between A and B is $d_H(A,B) := \max_{b\in B}\{\min_{a\in A}\operatorname{dist}(a,b)\}$.

Given a convex set C, there is a classic inner approximation $P_\varepsilon \subset C$ by Dudley [9] with $O(1/\sqrt{\varepsilon})$ vertices such that the Hausdorff-distance of P_ε and C is at most $\varepsilon d(C)$. Ahn et al. [1] showed that Dudley's method can be implemented in $O(T_C/\sqrt{\varepsilon})$ time.

The bound on the Hausdorff-distance guarantees that $C \setminus P_\varepsilon$ is "narrow." We will need the stronger property that every component of $C \setminus P_\varepsilon$ is small *in any direction.*

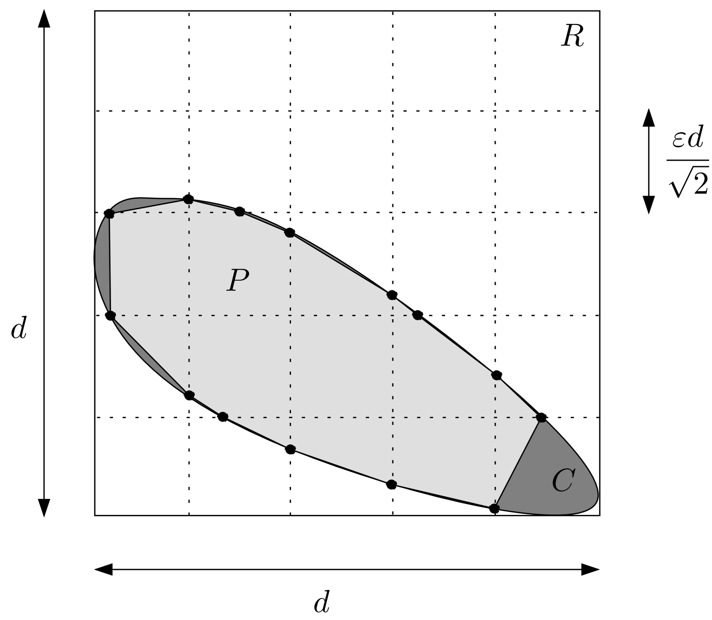

Fig. 8. Inner approximation of a convex shape.

Lemma 13 *Given a convex set C in the plane and $\varepsilon > 0$, one can construct in time $O(T_C/\varepsilon)$ a convex polygon $P \subset C$ with $O(1/\varepsilon)$ vertices such that any line ℓ intersects $C \setminus P$ in at most two segments of length at most $\varepsilon d(C)$.*

PROOF. We start by computing an axis-parallel square R circumscribed to C (that is, C touches two opposite sides of R), see Figure 8. This can be done using four extreme point queries. The side length of R is denoted by d, and $d \leqslant d(C)$. We then partition R with $2\sqrt{2}/\varepsilon$ equally spaced horizontal and vertical lines at a distance of $\varepsilon d/\sqrt{2}$, compute the intersection points of all these lines with $\mathrm{bd}(C)$, and let P be the convex hull of these points. Any connected component of $C \setminus P$ is contained in a square cell of diameter εd, implying the lemma. $\square$

While the lemma guarantees a stronger approximation, it needs far more vertices than Dudley's method. The bound of $\Theta(1/\varepsilon)$ vertices is tight, however, as can easily be seen by considering the case of a circle.

In our second approximation, we return to the Hausdorff-distance, but require that it be less than $\varepsilon w(C)$ (instead of $\varepsilon d(C)$ as guaranteed by Dudley's method). A similar lemma was already proven by Ahn et al. [1] for polygons. We give a proof for arbitrary convex sets C, and a formulation that makes it easy to bound the area of the difference $C \setminus P$.

Lemma 14 *Given a convex set C in the plane and $\varepsilon > 0$, one can construct in time $O(T_C/\varepsilon)$ a convex polygon $P \subset C$ with $O(1/\varepsilon)$ vertices and a line ℓ*

such that any line ℓ' parallel to ℓ intersects $C \setminus P$ in at most two segments of length at most $\varepsilon w(C)$.

PROOF. We first compute the rectangle R as in Lemma 4. We now use an orthonormal basis where the x-axis is parallel to the longer side of R. The boundary of C consists of two y-monotone chains C_l and C_r. We will select points on these chains to form the set of vertices of P. We select the lowest and highest point (that is, the point with smallest and largest y-coordinate), as well as the leftmost point of C_l and the rightmost point of C_r. We also ensure that any two consecutive vertices of P have vertical distance at most $\varepsilon w(C)$. This immediately implies that any vertical line intersects $C \setminus P$ in segments of length at most $\varepsilon w(C)$.

By Lemma 4, the shorter side of R has length at most $2\sqrt{2}w(C)$. We can therefore cover R by $2\sqrt{2}/\varepsilon$ horizontal lines equally spaced at distance $\varepsilon w(C)$. We compute the intersection points of these lines and $\mathrm{bd}(C)$, in time $O(T_C/\varepsilon)$. □

As before, this approximation requires $\Theta(1/\varepsilon)$ vertices, instead of the $\Theta(1/\sqrt{\varepsilon})$ vertices sufficient for Dudley's method. Again, the bound is tight, as the following lemma shows.

Lemma 15 *For all integers $n > 4$, there exists an n-gon P_n such that, for any k-gon Q contained in P_n with $d_H(Q, P_n) \leqslant w(P_n)/5n$, we have $k \geqslant (n-3)/2$.*

PROOF. Let v_i be the point with coordinates $(2^i, i)$, for integers $0 \leqslant i < n-1$, let $v_{n-1} = (2^{n-2}, 0)$, and let P_n be the convex hull of $\{v_i \mid 0 \leqslant i < n\}$. Note that the width of P_n is less than $n-2$. We assume that $Q \subset P_n$ is a convex k-gon such that $d_H(Q, P_n) \leqslant w(P_n)/5n$, so in particular $d_H(Q, P_n) < 1/5$.

For $0 < i < n-2$, let d_i denote the distance between the vertex v_i and the line segment $v_{i-1}v_{i+1}$. We observe that the distance between v_i and the point of the segment $v_{i-1}v_{i+1}$ with the same x-coordinate is $1/3$. Since the slope of $v_{i-1}v_{i+1}$ is less than 1, it follows that d_i is at least $1/(3\sqrt{2})$, so $d_i > 1/5$. Now suppose that no vertex of Q lies in the interior of some triangle $v_{i-1}v_iv_{i+1}$. Then the distance between v_i and Q is at least $d_i > 1/5$, a contradiction. Therefore, for all $0 < i < n-2$, there is a vertex of Q in the interior of the triangle $v_{i-1}v_iv_{i+1}$. The triangles $v_{i-1}v_iv_{i+1}$ where i is odd and $0 < i < n-2$ have disjoint interiors, and there are at least $(n-3)/2$ such triangles, so Q has at least $(n-3)/2$ vertices. □

5 Putting it all together

Our main theorem is the following.

Theorem 16 *Given two convex sets C_1 and C_2 in the plane and $\alpha > 0$, we can compute in time $O((T_{C_1} + T_{C_2})/\alpha + (1/\alpha^2)\log(1/\alpha))$ a rigid motion φ^{app} such that the area of $\varphi^{\mathrm{app}} C_1 \cap C_2$ is at least $1 - \alpha$ times the maximum over all rigid motions.*

PROOF. We start by computing circumscribed rectangles R_i for C_i according to Lemma 4, for $i = 1, 2$. Let $d_i \geqslant w_i$ be the sides of R_i. By Lemma 4 we have

$$\begin{aligned} d(C_i)/\sqrt{2} &\leqslant d_i \leqslant d(C_i), \\ w(C_i) &\leqslant w_i \leqslant 2\sqrt{2} w(C_i). \end{aligned}$$

Let $c := 3\sqrt{2}$. If $d_1 \geqslant cd_2$ and $w_1 \geqslant cw_2$, then we use Lemma 4 to compute a homothet r_1 of R_1 that is contained in C_1 and has side lengths $d_1/c \geqslant d_2$ and $w_1/c \geqslant w_2$. In constant time we can find a rigid motion φ such that $R_2 \subset \varphi r_1$. It follows that $C_2 \subset \varphi C_1$, so φ is an optimal solution with value $|C_2| = \min(|C_1|, |C_2|)$.

We can handle the case where $d_2 \geqslant cd_1$ and $w_2 \geqslant cw_1$ in the same way, so in the following, we assume that ($d_2 < cd_1$ or $w_2 < cw_1$) and ($d_1 < cd_2$ or $w_1 < cw_2$). Without loss of generality, we assume that $d_2/w_2 \geqslant d_1/w_1$ (otherwise we swap C_1 and C_2). This implies that $w_2 \leqslant cw_1$ (if $w_2 > cw_1$ then $d_2 < cd_1$, which implies $d_2/w_2 < d_1/w_1$) and $d_1 \leqslant cd_2$ (if $d_1 > cd_2$ then $w_1 < cw_2$, which implies again $d_2/w_2 < d_1/w_1$). Therefore $d_1 w_2 \leqslant d_2 w_1$, $d_1 w_2 \leqslant cd_2 w_2$, and $d_1 w_2 \leqslant d_1 c w_1$, so

$$\begin{aligned} d_1 w_2 &\leqslant c \min\{d_1, d_2\} \min\{w_1, w_2\} \\ &\leqslant 2c\sqrt{2} \min\{d(C_1), d(C_2)\} \min\{w(C_1), w(C_2)\}. \end{aligned}$$

Choosing $\varepsilon = \alpha/(128c + 1)$, we compute the approximation P_1 of Lemma 13 for C_1, the approximation P_2 of Lemma 14 for C_2, and then compute the rigid motion φ^{app} of Lemma 12 such that

$$|\varphi^{\mathrm{app}} P_1 \cap P_2| \geqslant (1 - \varepsilon) \max_{\varphi} |\varphi P_1 \cap P_2|.$$

The total running time is $O((T_{C_1} + T_{C_2})/\varepsilon + (1/\varepsilon^2)\log(1/\varepsilon))$, and since $\alpha = \Theta(\varepsilon)$, it is also $O((T_{C_1} + T_{C_2})/\alpha + (1/\alpha^2)\log(1/\alpha))$. So it only remains to prove that this choice of φ^{app} provides the desired approximation.

Let φ^{opt} be a rigid motion such that

$$|\varphi^{\text{opt}} C_1 \cap C_2| = \max_{\varphi} |\varphi C_1 \cap C_2|.$$

We will show below that, for any rigid motion φ,

$$|\varphi P_1 \cap P_2| \geqslant |\varphi C_1 \cap C_2| - 128c\varepsilon |\varphi^{\text{opt}} C_1 \cap C_2|. \tag{1}$$

We then have

$$\begin{aligned}\max_{\varphi} |\varphi P_1 \cap P_2| &\geqslant |\varphi^{\text{opt}} P_1 \cap P_2| \\ &\geqslant |\varphi^{\text{opt}} C_1 \cap C_2| - 128c\varepsilon |\varphi^{\text{opt}} C_1 \cap C_2|.\end{aligned}$$

By Lemma 12, we have

$$\begin{aligned}|\varphi^{\text{app}} C_1 \cap C_2| &\geqslant |\varphi^{\text{app}} P_1 \cap P_2| \\ &\geqslant (1-\varepsilon) \max_{\varphi} |\varphi P_1 \cap P_2| \\ &= \max_{\varphi} |\varphi P_1 \cap P_2| - \varepsilon \max_{\varphi} |\varphi P_1 \cap P_2| \\ &\geqslant \max_{\varphi} |\varphi P_1 \cap P_2| - \varepsilon \max_{\varphi} |\varphi C_1 \cap C_2| \\ &= \max_{\varphi} |\varphi P_1 \cap P_2| - \varepsilon |\varphi^{\text{opt}} C_1 \cap C_2| \\ &\geqslant |\varphi^{\text{opt}} C_1 \cap C_2| - 128c\varepsilon |\varphi^{\text{opt}} C_1 \cap C_2| \\ &\quad -\varepsilon |\varphi^{\text{opt}} C_1 \cap C_2| \\ &= |\varphi^{\text{opt}} C_1 \cap C_2| - (128c+1)\varepsilon |\varphi^{\text{opt}} C_1 \cap C_2| \\ &= (1-\alpha) |\varphi^{\text{opt}} C_1 \cap C_2|.\end{aligned}$$

It remains to prove the claim (1) above for a rigid motion φ.

Let $D_1 := (\varphi C_1 \setminus \varphi P_1) \cap C_2$. By Lemma 13, any line ℓ parallel to the longer side of R_2 intersects D_1 in at most two segments of length at most $\varepsilon d(C_1)$. Integrating over the shorter side of R_2, we find $|D_1| \leqslant 2\varepsilon w_2 d(C_1)$.

Let $D_2 := (C_2 \setminus P_2) \cap \varphi C_1$. By Lemma 14, there is a line ℓ such that any line ℓ' parallel to ℓ intersects D_2 in at most two segments of length at most $\varepsilon w(C_2) \leqslant \varepsilon w_2$. Since $d(D_2) \leqslant d(C_1)$, it suffices to integrate over an interval of length $d(C_1)$ to obtain $|D_2| \leqslant 2\varepsilon w_2 d(C_1)$.

Since $(\varphi C_1 \cap C_2) \setminus (\varphi P_1 \cap P_2) \subset D_1 \cup D_2$, we have

$$
\begin{aligned}
|(\varphi C_1 \cap C_2) \setminus (\varphi P_1 \cap P_2)| &\leqslant |D_1 \cup D_2| \leqslant |D_1| + |D_2| \\
&\leqslant 4\varepsilon w_2 d(C_1) \leqslant 4\sqrt{2}\varepsilon d_1 w_2 \\
&\leqslant 4\sqrt{2}\varepsilon 2c\sqrt{2} \\
&\quad \times \min(w(C_1), w(C_2)) \\
&\quad \times \min(d(C_1), d(C_2)) \\
&\leqslant 128c\varepsilon |\varphi^{\mathrm{opt}} C_1 \cap C_2|.
\end{aligned}
$$

The last inequality is due to Lemma 6. This implies $|\varphi P_1 \cap P_2| \geqslant |\varphi C_1 \cap C_2| - 128c\varepsilon|\varphi^{\mathrm{opt}} C_1 \cap C_2|$, completing the proof. $\square$

6 Translations only

With the same techniques, we can also handle the case where the rigid motion is restricted to be a translation τ. The key idea is to first apply an affine transformation that makes C_2 fat. Once we have that, we apply the approximation of Lemma 13 to C_2 and the approximation of Lemma 14 to C_1. This does not decrease the area of overlap by more than α of the optimum, and so we can finally use Lemma 9 on the approximations. The proof is very similar to the proof of Theorem 16.

Theorem 17 *Given two convex sets C_1 and C_2 in the plane and $\alpha > 0$, we can compute in time $O((T_{C_1} + T_{C_2})/\alpha + (1/\alpha)\log(1/\alpha))$ a translation τ^{app} such that the area of $\tau^{\mathrm{app}} C_1 \cap C_2$ is at least $1 - \alpha$ times the maximum over all translations.*

PROOF. We first compute the two rectangles $r_2 \subset C_2 \subset R_2$ of Lemma 4. There is an affine transformation f that maps r_2 to the unit square. Since f preserves area ratios, our problem is equivalent to finding an approximate maximum overlap of $f(C_1)$ and $f(C_2)$ under translation. So in the remainder of this proof, we will assume, without loss of generality, that $r_2 \subset C_2 \subset R_2$ where r_2 is the unit square and R_2 is an axis–parallel square with side length $3\sqrt{2}$.

We now compute the two homothetic rectangles $r_1 \subset C_1 \subset R_1$ of Lemma 4. We denote by $w_1 \leqslant d_1$ the lengths of the sides of R_1.

If $d_1 \leqslant 1/\sqrt{2}$, then we can find in constant time a translation τ such that $\tau R_1 \subset r_2$. Therefore $\tau C_1 \subset C_2$, and we can choose $\tau^{\mathrm{app}} = \tau$. So from now on we assume that $d_1 > 1/\sqrt{2}$, which implies $d(C_1) > 1/\sqrt{2}$.

On the other hand, suppose that $w_1 \geqslant 18\sqrt{2}$. Then the shorter side of r_1 has length $w_1/(3\sqrt{2}) \geqslant 6$. So we can find in constant time a translation τ such that $R_2 \subset \tau r_1$. Therefore $C_2 \subset \tau C_1$, and we can again choose $\tau^{\text{app}} = \tau$. So from now on we assume that $w_1 < 18\sqrt{2}$, which implies $w(C_1) < 18\sqrt{2}$.

We fix $\varepsilon = \alpha/(432\sqrt{2})$. We apply Lemma 14 to C_1, so we obtain an ε–approximating polygon P_1. We apply Lemma 13 to C_2 and obtain an ε–approximating polygon P_2. Notice that until now we have spent only $O((T_{C_1} + T_{C_2})/\alpha)$ time. Now we apply Lemma 9 and find the translation τ^{app} that maximizes $|\tau^{\text{app}} P_1 \cap P_2|$. This takes time $O((1/\alpha)\log(1/\alpha))$. Let τ^{opt} be a rigid motion such that

$$|\tau^{\text{opt}} C_1 \cap C_2| = \max_{\tau} |\tau C_1 \cap C_2|.$$

We will show below that, for any translation τ,

$$|\tau P_1 \cap P_2| \geqslant |\tau C_1 \cap C_2| - \alpha |\tau^{\text{opt}} C_1 \cap C_2|. \tag{2}$$

We then have

$$\begin{aligned}|\tau^{\text{app}} C_1 \cap C_2| &\geqslant |\tau^{\text{app}} P_1 \cap P_2| \geqslant |\tau^{\text{opt}} P_1 \cap P_2| \\ &\geqslant |\tau^{\text{opt}} C_1 \cap C_2| - \alpha |\tau^{\text{opt}} C_1 \cap C_2|,\end{aligned}$$

and thus we proved that τ^{app} provides the desired approximation.

It remains to prove Claim (2). By Lemma 5, C_1 contains a rectangle r with sides $d(C_1)/2$ and $w(C_1)/4$. If $w(C_1) \geqslant 4\sqrt{2}$, then both sides of r have length at least $\sqrt{2}$, and so there is a translation that maps r such that it covers r_2. This implies that

$$|\tau^{\text{opt}} C_1 \cap C_2| \geqslant |r_2| = 1 \geqslant \frac{w(C_1)}{18\sqrt{2}}.$$

If, on the other hand, $w(C_1) < 4\sqrt{2}$, then we consider a smaller rectangle r' with sides $1/2\sqrt{2}$ and $w(C_1)/8$ contained in r. Since both sides of r' have length at most $1/\sqrt{2}$, there is a translation that maps r' inside r_2, which implies again that

$$|\tau^{\text{opt}} C_1 \cap C_2| \geqslant |r'| = \frac{1}{2\sqrt{2}} \times \frac{w(C_1)}{8} > \frac{w(C_1)}{18\sqrt{2}} \tag{3}$$

Let $D_1 := (\tau C_1 \setminus \tau P_1) \cap C_2$. By Lemma 14, there is a line ℓ such that any line ℓ' parallel to ℓ intersects $\tau C_1 \setminus \tau P_1$ in at most two segments of length at most $\varepsilon w(C_1)$. Since $d(C_2) \leqslant 6$, it suffices to integrate over an interval of length $d(C_2)$ to obtain $|D_1| \leqslant 12\varepsilon w(C_1)$.

Let $D_2 := (C_2 \setminus P_2) \cap \tau C_1$. Let R be a bounding rectangle of τC_1 with one side length equal to $w(C_1)$. By Lemma 13, any line ℓ parallel to the longer side of R intersects D_1 in at most two segments of length at most $\varepsilon d(C_2) \leqslant 6\varepsilon$. Integrating over the shorter side of R, we find $|D_2| \leqslant 12\varepsilon w(C_1)$.

Since $(\tau C_1 \cap C_2) \setminus (\tau P_1 \cap P_2) \subset D_1 \cup D_2$, we have

$$|(\tau C_1 \cap C_2) \setminus (\tau P_1 \cap P_2)| \leqslant |D_1 \cup D_2| \leqslant 24\varepsilon w(C_1).$$

By Equation (3), it yields

$$|(\tau C_1 \cap C_2) \setminus (\tau P_1 \cap P_2)| \leqslant \alpha|\tau^{\mathrm{opt}} C_1 \cap C_2|,$$

which completes the proof of Claim (2). $\square$

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