Empty Pseudo-Triangles in Point Sets*

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Abstract

We study empty pseudo-triangles in a set P of n points in the plane, where an empty pseudo-triangle has its three convex vertices and its concave vertices at the points of P, and no points of P lie inside. We give bounds on the number of possible empty pseudo-triangles. If the three convex vertices are fixed, there can be between $\Theta(n^2)$ and $\Theta(n^3)$ empty pseudo-triangles, whereas if the convex vertices are not fixed, this number lies between $\Theta(n^3)$ and $\Theta(n^6)$. If we count only star-shaped pseudo-triangles, the bounds are $\Theta(n^2)$ and $\Theta(n^5)$. We also study optimization problems: minimizing or maximizing the perimeter or the area over all empty pseudo-triangles defined by P. If the convex vertices are fixed, we can solve these problems in $O(n^3)$ time. If the convex vertices are not given, the running times are $O(n^6)$ for the maximization problems and $O(n \log n)$ for the minimization problems.

1 Introduction

A *pseudo-triangle* is a simple polygon with exactly three convex vertices. These convex vertices are connected by straight line segments or by chains of concave vertices (we consider a vertex with internal angle π to be concave). By definition, any triangle is a pseudo-triangle, and the convex hull of any pseudo-triangle is a triangle, see Figure 1. A pseudo-triangle is *star-shaped* if a point q exists in the interior such that for any point p in the pseudo-triangle, the whole line segment $\overline{pq}$ is inside as well.

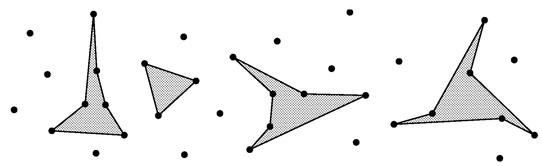

Figure 1: Empty pseudo-triangles in a point set.

Pseudo-triangles were introduced in the context of computing visibility relations among convex obstacles in the plane [15, 16]. Later, a number of different optimization problems of *pseudo-triangulations*, i.e., decompositions of a region into pseudo-triangles, have been studied [1, 3, 12, 14, 17]. For an overview of pseudo-triangulations we refer to the survey by Rote et al. [18]. In this paper, we are interested in empty pseudo-triangles defined by a given set P of n points in the plane. A pseudo-triangle is defined by a set of points if its vertices are taken from the set, and it is empty if no points of the set lie in the interior.

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Table 1: Bounds on the number of empty pseudo-triangles in a point set.

Type of PT	Fixed convex corners		Arbitrary corners	
	minimum	maximum	minimum	maximum
general	$\Theta(n^2)$	$\Theta(n^3)$	$\Theta(n^3)$	$\Theta(n^6)$
star-shaped	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^3)$	$\Theta(n^5)$

Table 2: Running times for optimization problems on empty pseudo-triangles in a point set.

Objective	Fixed convex corners	Arbitrary corners
minimize perimeter or area	$O(n^3)$	$O(n \log n)$
maximize perimeter or area	$O(n^3)$	$O(n^6)$

Counting empty convex k-gons defined by planar point sets is a classic problem in combinatorial geometry that goes back to Erdős [11]. In particular, he asked for the smallest number $N(k)$ such that any set P of at least $N(k)$ points contains the vertex set of a convex k-gon that does not contain any point of P in its interior. A related question asks for the minimum number of empty convex k-gons any set of n points must contain. It is typically assumed that the n points lie in *general position*: no three points are co-linear.

Concerning the number of empty convex polygons, it was shown in [5, 9, 13] that for any set P of n points in general position, there are $\Omega(n^2)$ subsets of three, four, five, and six points that form empty convex triangles, quadrilaterals, pentagons, and hexagons. These bounds are tight. Furthermore, there are arbitrarily large sets of points that do not contain any empty convex heptagon. Trivially, for any constant k, the maximum number of empty convex k-gons is $\Theta(n^k)$, which is obtained by taking n points in convex position.

A related algorithmic question is to ask for smallest or largest shapes defined by a set P of n points. Edelsbrunner and Guibas showed that the minimum area triangle (which is necessarily empty) with its vertices at points of P can be computed in $O(n^2)$ time [10]. The largest empty circle in a bounded region containing n points can be computed in $O(n \log n)$ time using generalized Voronoi diagrams [8, 19]. The largest empty rectangle in a bounded region can be determined in $O(n^3)$ time [6].

In this paper we first determine combinatorial bounds on the minimum and maximum number of pseudo-triangles that are defined by a set P of n points. If we do not require the pseudo-triangles to be empty of points in P, then there can be exponentially many. For example, place one point p_0 at the origin and all other $n-1$ points on the lower left quarter of the circle $(x-1)^2+(y-1)^2 = 1$. Then p_0 together with any subset of $P \setminus \{p_0\}$ of size ≥ 2 forms a pseudo-triangle, so there are at least $2^{n-1} - n$ of them. On the other hand, the minimum number of pseudo-triangles is cubic (e.g., for points in convex position, see Theorem 6).

We focus on empty pseudo-triangles, and first prove some observations in Section 2. In Section 3.1 we assume that the three convex vertices of the pseudo-triangle are fixed, and there are n points inside the triangle defined by the convex vertices. We analyze the number of empty pseudo-triangles in this case. We study four combinatorial questions, namely the *minimum* and *maximum* number of empty *general* and *star-shaped* pseudo-triangles. We give tight upper and lower bounds for each question; our results are summarized in the left half of Table 1. Observe that the (asymptotic) number of empty pseudo-triangles in the general case can be quadratic or cubic, depending on the point set, but there are always quadratically many star-shaped pseudo-triangles. If the convex vertices of the empty pseudo-triangles are not fixed, then we get the same four questions, and most of the ideas used before can be extended. We obtain the results summarized in the right half of Table 1.

We consider the following four optimization problems in Section 4: minimizing or maximizing the perimeter, and minimizing or maximizing the area over all empty pseudo-triangles. Again,

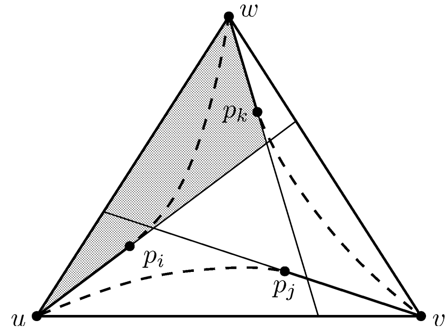

Figure 2: Every empty pseudo-triangle is uniquely determined by a triple (i, j, k) of indices.

we study these problems when the convex vertices of the pseudo-triangle are fixed and when they can be freely chosen among the points of P. The running times of our results are summarized in Table 2, the space requirement is $O(n)$ in all cases. The efficiency of the minimization problems in the general case follows from the observation that the smallest pseudo-triangle will be a triangle, which simplifies the problem drastically.

2 Preliminaries

Let u, v, and w be three points in the plane. Without loss of generality, we assume that the segment connecting u and v is horizontal with u to the left of v, and that w lies above the segment. Let P be a set of n points inside the triangle $\triangle uvw$, and we assume that no three points of $P \cup \{u, v, w\}$ lie on a line. We denote the line that passes through two points p and q by $\ell(p, q)$.

Any pseudo-triangle with u, v, and w as the convex vertices has a concave chain between pairs of these vertices. Each convex vertex is the common point of two concave chains. Let p_i, p_j, and p_k be the first vertices encountered when we follow the concave chains clockwise along the pseudo-triangle from u, v, and w, respectively. Figure 2 shows these vertices and three lines, each induced by a corner and its corresponding vertex. Consider the gray region bounded by the lines $\ell(u, w)$, $\ell(u, p_i)$ and $\ell(w, p_k)$. For the pseudo-triangle to be empty, all points from P inside the gray region must be enclosed by the concave chain connecting u and w. Therefore, the polygonal chain enclosing the points in the gray region is uniquely defined by p_i and p_k, and we denote it by $C_{uw}(i, k)$. Analogously, the concave chain $C_{vw}(k, j)$ connecting w and v is uniquely defined by p_k and p_j, and the concave chain $C_{uv}(j, i)$ connecting v and u is uniquely defined by p_j and p_i. We observe:

Observation 1 *Three points p_i, p_j, and p_k define an empty pseudo-triangle if and only if:*
(i) point p_i lies left of $\ell(w, p_k)$, point p_j lies below $\ell(u, p_i)$, and point p_k lies above $\ell(v, p_j)$, and
(ii) the triangular region below $\ell(u, p_i)$, above $\ell(v, p_j)$ and left of $\ell(w, p_k)$ does not contain any point of P in its interior.
Furthermore, there is at most one such pseudo-triangle.

In the following, we simplify the notation of the concave chains by omitting the indices of the chains if the points p_i, p_j, and p_k are defined in the context. The chains are then denoted by C_{uw}, C_{vw}, and C_{uv}.

We assume all points of the input are in general position. In particular, no three points lie on a line.

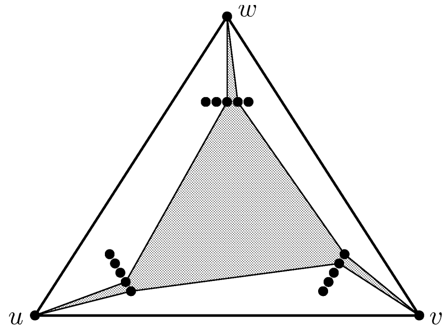

Figure 3: Construction for the $\Omega(n^3)$ lower bound on the maximum number of empty pseudo-triangles.

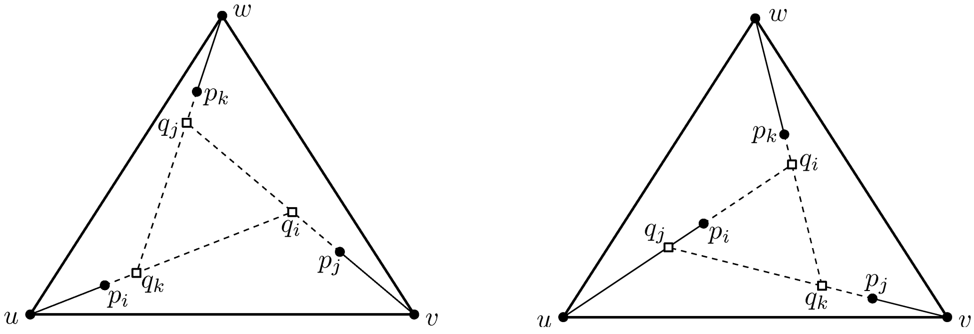

Figure 4: Construction for the $O(n^3)$ upper bound on the maximum number of empty pseudo-triangles.

3 The Number of Empty Pseudo-Triangles

3.1 Pseudo-triangles with given corners

In this section we give tight upper and lower bounds on the number of empty pseudo-triangles. First, we assume that the convex vertices of the pseudo-triangle are fixed, and we show that there are different upper bounds if we allow any empty pseudo-triangle and if we restrict ourselves to only star-shaped pseudo-triangles. Second, we drop the assumption of fixed convex vertices, which leads to an increase in both bounds on both types of empty pseudo-triangles.

3.1.1 General pseudo-triangles

Theorem 1 *Given three points u, v, w and a set P of n points inside $\triangle uvw$, the maximum number of empty pseudo-triangles with u, v, w as the convex vertices is $\Theta(n^3)$.*

Proof: The lower bound is an easy construction, see Figure 3.

To prove the upper bound, let p_i, p_j, and p_k be any three points of P. We analyze the number of (empty) pseudo-triangles such that edge $\overline{up_i}$ is on the chain C_{uw}, edge $\overline{vp_j}$ is on the chain C_{uv}, and edge $\overline{wp_k}$ is on the chain C_{vw}. Clearly, if any two of the three edges intersect, then no pseudo-triangle of this type exists. Otherwise, we extend the edges $\overline{up_i}$, $\overline{vp_j}$, and $\overline{wp_k}$ in a special way.

Assume first that $\ell(u, p_i)$ is below $\ell(v, p_j) \cap \ell(w, p_k)$. Then we let point $q_i = \ell(u, p_i) \cap \ell(v, p_j)$, point $q_j = \ell(v, p_j) \cap \ell(w, p_k)$, and point $q_k = \ell(w, p_k) \cap \ell(u, p_i)$, see Figure 4 (left).

If $\triangle q_i q_j q_k$ contains points of P, then no pseudo-triangle of this type exists: Either the corresponding pseudo-triangle is not empty, or the concavity of one of the chains is compromised. Furthermore, all points in $\triangle uvq_i$ must be excluded via the chain C_{uv}, all points in $\triangle vwq_j$ must

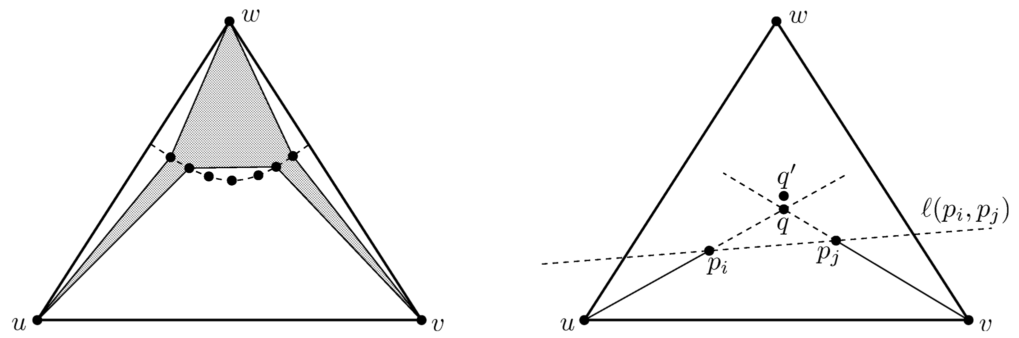

Figure 5: Construction for the $O(n^2)$ upper bound (left) and $\Omega(n^2)$ lower bound (right) of the minimum number of empty pseudo-triangles.

be excluded via the chain C_{vw}, and all points in $\triangle wuq_k$ must be excluded via the chain C_{uw}. This fully defines the partition as in Observation 1, so we count at most one pseudo-triangle.

Next assume that $\ell(u, p_i)$ is above $\ell(v, p_j) \cap \ell(w, p_k)$; note that due to the specification of p_i, p_j, and p_k, this case is not symmetric to the previous one. The argument, however, is still analogous. This time we let point $q_i = \ell(u, p_i) \cap \ell(w, p_k)$, point $q_j = \ell(v, p_j) \cap \ell(u, p_i)$, and point $q_k = \ell(w, p_k) \cap \ell(v, p_j)$, see Figure 4 (right). If $\triangle q_i q_j q_k$ contains points of P, then no pseudo-triangle of this type exists: Either the corresponding pseudo-triangle is not empty, or the concavity of one of the chains is compromised. The argument is exactly as in the previous case, so we again count at most one pseudo-triangle.

All remaining cases ($\ell(u, p_i)$ contains $\ell(v, p_j) \cap \ell(w, p_k)$, or some chain(s) do not contain points of P) are straightforward to analyze. Since there are $6 \cdot \binom{n}{3}$ choices for p_i, p_j, and p_k, the upper bound follows. $\square$

Theorem 2 *Given three points u, v, w and a set P of n points inside $\triangle uvw$, the minimum number of empty pseudo-triangles with u, v, w as the convex vertices is $\Theta(n^2)$.*

Proof: This time we begin with the upper bound, which is an easy construction shown in Figure 5 (left). All points of P are placed on a circular arc centered at w. There are only $O(n)$ choices for the chain C_{vw}, only $O(n)$ choices of the chain C_{uw}, and given these choices, the chain C_{uv} is completely specified since we count only empty pseudo-triangles. The quadratic upper bound follows.

Next we prove the lower bound. We need to show that any set P gives $\Omega(n^2)$ (empty) pseudo-triangles. Let (p_i, p_j) be any pair of points from P. If $\ell(p_i, p_j)$ does not intersect $\overline{uv}$ then we assign the pair to $\overline{uv}$. Similarly, if $\ell(p_i, p_j)$ does not intersect $\overline{vw}$ then we assign the pair to $\overline{vw}$, and if $\ell(p_i, p_j)$ does not intersect $\overline{uw}$ then we assign the pair to $\overline{uw}$. Due to non-degeneracy, we assign each pair from P to exactly one side of $\triangle uvw$.

By symmetry and the pigeon-hole principle we may assume that $\Omega(n^2)$ pairs of points are assigned to $\overline{uv}$. Make each pair ordered so that $\overline{up_i}, \overline{p_i p_j}, \overline{p_j v}$ is a concave chain that does not self-intersect. Let $q = \ell(u, p_i) \cap \ell(v, p_j)$, see Figure 5 (right), and let q' be infinitesimally above q. Then an empty pseudo-triangle exists that excludes the points of $P \cap \triangle uvq'$ via the chain C_{uv}, that excludes the points of $P \cap \triangle vwq'$ via the chain C_{vw}, and that excludes the points of $P \cap \triangle uwq'$ via the chain C_{uw}. Furthermore, $\overline{up_i}$ and $\overline{vp_j}$ are the extreme edges of C_{uv}. Hence, for any other pair (p_l, p_m) assigned to $\overline{uv}$ we get a different pseudo-triangle. Since $\Omega(n^2)$ edges were assigned to $\overline{uv}$, there are $\Omega(n^2)$ different pseudo-triangles. $\square$

Note that the non-degeneracy assumption made previously is essential. If all points of P lie on a line that also passes through one convex vertex, then there are only $O(n)$ different empty pseudo-triangles.

3.1.2 Star-shaped pseudo-triangles

Interestingly, the number of empty star-shaped pseudo-triangles does not vary with P, asymptotically, and is always quadratic. The lower-bound proof of Theorem 2 generates only star-shaped pseudo-triangles, because q' is always in the kernel. So the *minimum* number of empty shar-shaped pseudo-triangles is $\Omega(n^2)$. It remains to prove that the *maximum* number of empty star-shaped pseudo-triangles is also $O(n^2)$.

Lemma 1 *Given three points u, v, w and a set P of n points inside $\triangle uvw$, the maximum number of empty star-shaped pseudo-triangles with u, v, w as the convex vertices is $O(n^2)$.*

Proof: Consider the set of $3n$ lines defined by one point of P and one point of $\{u, v, w\}$, see Figure 6. These lines form an arrangement of quadratic size. Let q be any point inside a cell

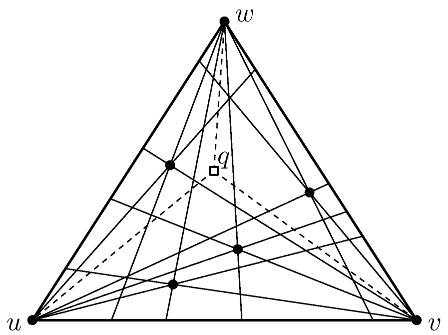

Figure 6: Always $O(n^2)$ star-shaped pseudo-triangles.

of the arrangement. If we assume that q is in the kernel of the empty pseudo-triangle, then the pseudo-triangle is completely determined: all points of $P \cap \triangle uvq$ are excluded via the chain C_{uv}, and the analogous statement holds for $P \cap \triangle vwq$ and $P \cap \triangle uwq$. By the choice of lines, no matter where q lies in its cell, the subsets $P \cap \triangle uvq$, $P \cap \triangle vwq$, and $P \cap \triangle uwq$ are the same. Since there are $O(n^2)$ combinatorially distinct positions for q, the lemma follows. $\square$

Theorem 3 *Given three points u, v, w and a set P of n points inside $\triangle uvw$, the minimum and maximum number of empty star-shaped pseudo-triangles with u, v, w as convex vertices is $\Theta(n^2)$.*

3.2 Pseudo-triangles in point sets

We discuss the case where the convex vertices are not given in advance. The results in the previous section give rise to some easy results for this case.

Theorem 4 *Given a set P of n points in the plane, the maximum number of empty pseudo-triangles is $\Theta(n^6)$.*

Proof: The upper bound follows by taking all triples of P as u, v, and w and using the result of Theorem 1.

The lower bound follows by taking the construction of Theorem 1, using only $n/2$ points inside $\triangle uvw$, and replacing u, v, and w by $n/6$ points each. w is replaced by $n/6$ points on a horizontal line, very closely spaced and at w. Similarly, u and v are replaced by $n/6$ points each, and on lines that make angles of 60 degrees (for v) and -60 degrees (for u) with the x-axis. $\square$

The same proof adaptations give the result on the maximum number of star-shaped pseudo-triangles. We simply state the result:

Theorem 5 *Given a set P of n points in the plane, the maximum number of empty star-shaped pseudo-triangles is $\Theta(n^5)$.*

We will give one more result, namely that any point set gives $\Omega(n^3)$ different empty star-shaped pseudo-triangles. It completes the study of the number of empty pseudo-triangles, since a point set in convex position gives only $O(n^3)$ different pseudo-triangles.

Theorem 6 *Given a set P of n points in the plane, the minimum number of empty pseudo-triangles (star-shaped or arbitrary) is $\Theta(n^3)$.*

Proof: We need to prove only two results: there is a set of n points that gives $O(n^3)$ empty (not necessarily star-shaped) pseudo-triangles, and any point set gives $\Omega(n^3)$ empty star-shaped pseudo-triangles. For the former claim, simply take a set of n points in convex position. For the latter claim, take any three points p_i, p_j, and p_k of P. We will show that an empty star-shaped pseudo-triangle exists with p_i, p_j, and p_k as the convex vertices. Take any point q (not from the input) in the interior of $\triangle p_i p_j p_k$, and so that $\overline{p_i q}$, $\overline{p_j q}$, and $\overline{p_k q}$ do not contain any point of P besides p_i, p_j and p_k themselves. Consider the pseudo-triangle that excludes any points of $P \cap \triangle p_i p_j q$ via chain C_{p_i,p_j}, any points of $P \cap \triangle p_j p_k q$ via chain C_{p_j,p_k}, and any points of $P \cap \triangle p_k p_i q$ via chain C_{p_k,p_i}. Clearly this gives a pseudo-triangle with p_i, p_j, and p_k as the convex vertices and q in the kernel. All $\binom{n}{3}$ choices of p_i, p_j, and p_k give different pseudo-triangles. $\square$

4 Computing Optimal Empty Pseudo-Triangles

In this section we study the algorithmic problem of computing an empty pseudo-triangle that is optimal with respect to its perimeter or its area. We consider both minimization and maximization for each. As before, we first discuss the case where the three convex vertices are fixed, and then we proceed with the case where they are not.

4.1 Optimal empty pseudo-triangles with given corners

Given three points u, v, and w and a set P of n points inside $\triangle uvw$, we show how to determine an optimal pseudo-triangle that has u, v, and w as its convex vertices, its other vertices at points of P, and no points of P inside.

Recall from the proof of Theorem 1 that three points $p_i, p_j, p_k \in P$ define at most one empty pseudo-triangle, if we assume that the edges $\overline{up_i}$, $\overline{vp_j}$ and $\overline{wp_k}$ are the first edges of the chains C_{uv}, C_{vw}, and C_{wu}, respectively. This observation immediately leads to an $O(n^4)$ time algorithm: Presort the points by x-coordinate, enumerate over all triples p_i, p_j, and p_k, and determine if an empty pseudo-triangle exists for that triple, using the cases from the proof of Theorem 1. The pseudo-triangle for each triple can be generated in linear time with a Graham scan convex hull algorithm [8], due to presorting. Also due to presorting we can determine emptiness in linear time by a simultaneous sweep over the point set and the pseudo-triangle. Finally, the area or perimeter can be computed in linear time as well.

We can improve this method to run in $O(n^3)$ time by enumerating all possibilities in a clever way. Essentially, we will fix only p_i and p_j, and handle all possibilities for p_k in linear time altogether. This is possible because the total summed change of consecutive pseudo-triangles that we test is only linear instead of quadratic.

Let us fix p_i and p_j and assume that $\overline{up_i}$ is an edge of C_{uw} and $\overline{vp_j}$ is an edge of C_{vw} (observe that now p_j is on C_{vw} and not on C_{uv}, as in the proof of Theorem 1). All points of P strictly below both $\ell(u, p_i)$ and $\ell(v, p_j)$ must be on or below the chain C_{uv}; hence, C_{uv} is determined completely. Let P' be the subset of points of P that lie above or on $\ell(u, p_i)$ or $\ell(v, p_j)$. Assume that their counterclockwise sorted order around w is $p_1, \ldots, p_m$. For any empty pseudo-triangle with p_i and p_j as specified, a point p_k exists such that all points left of or on $\ell(w, p_k)$ are on or left of C_{uw}, and all points right of or on $\ell(w, p_{k+1})$ are on or right of C_{vw}. This empty pseudo-triangle includes the edges $\overline{wp_k}$ on C_{uw} and $\overline{wp_{k+1}}$ on C_{vw}. Observe that $i \leq k < k+1 \leq j$, otherwise the pseudo-triangle is not empty, not convex, or self-intersecting. Furthermore, if p_i lies above $\ell(v, p_j)$ or p_j lies above $\ell(u, p_i)$, then a certain triangular region inside $\triangle uvw$ must be empty of points in P, otherwise no empty pseudo-triangle of the specified type can be constructed. We conclude that there are at most linearly many choices for the chain C_{uw} that include $\overline{up_i}$, and C_{uw} is determined if we fix p_k.

In the code below, we use $\ell(w,p)^{\text{left}}$ and $\ell(w,p)^{\text{right}}$ to denote the half-plane to the left or right of $\ell(w,p)$, respectively. We use $\ell(u,p)^-$, $\ell(u,p)^+$, $\ell(v,p)^-$, and $\ell(v,p)^+$ for the half-planes below or above the corresponding lines.

Algorithm *Test-Pair*
Input: A set P of n points inside $\triangle uvw$, and two points $p_i, p_j \in P$.
Output: TRUE if and only if some empty pseudo-triangle exists with $\overline{up_i}$ as an edge of C_{uw} and $\overline{vp_j}$ as an edge of C_{vw}.

if $\ell(w,p_i) \cap \overline{vp_j} \neq \emptyset$ or $\ell(w,p_j) \cap \overline{up_i} \neq \emptyset$ **return** FALSE
if $\ell(u,p_i)^- \cap \ell(v,p_j)^+ \cap \ell(w,p_i)^{\text{left}} \cap P \neq \emptyset$ **return** FALSE
if $\ell(u,p_i)^+ \cap \ell(v,p_j)^- \cap \ell(w,p_j)^{\text{right}} \cap P \neq \emptyset$ **return** FALSE
return TRUE

We consider all choices of p_k with $k \geq i$ and the concave chains C_{uw} they determine. The union of these chains is a graph G of different concave paths that connect u and w. The vertices of G include u, w, and the points of P' that lie inside the wedge above $\ell(u,p_i)$ and to the right of $\ell(w,p_i)$.

Lemma 2 *If we remove w and all edges incident to it from G, we obtain a planar embedding of a tree.*

Proof: Take G, remove w and all incident edges, and direct all remaining edges towards u. Let $\vec{e}$ be any edge of G. The directed supporting line of $\vec{e}$ has u and w to its right, otherwise the chain that includes $\vec{e}$ is not concave. Assume that two edges $\vec{e}$ and $\vec{f}$ of G exist that intersect. Assume without loss of generality that $\vec{f}$ intersects $\vec{e}$ from $\vec{e}$'s right to $\vec{e}$'s left. Then any concave chain C_{uw} that includes $\vec{e}$ cannot have the destination endpoint of $\vec{f}$ on it or to its left (recall that chains are undirected). Hence, this endpoint of $\vec{f}$ must be on or right of C_{vw}. But then the chains C_{uw} and C_{vw} must intersect (using the fact that w is to the right of the directed supporting line of $\vec{e}$). $\square$

We denote the tree referred to in the lemma above by $T_i(u)$, and we let u be the root. It can be computed with a simple incremental algorithm, see the left part of Figure 7:

Algorithm *Rooted-Tree*
Input: Vertex u, point set P' sorted counterclockwise around w, and point $p_i \in P'$.
Output: A tree $T_i(u)$.

$T_i(u) \leftarrow \overline{up_i}$
$p \leftarrow u$ and $q \leftarrow p_i$
$k \leftarrow i+1$
while $k < j$ **and** p_k is above $\ell(u,p_i)$
    Let γ be the ray directed from p towards q
    while p_k is to the right of γ
        $q \leftarrow p$, and $p \leftarrow$ the parent of p in $T_i(u)$
        γ is the ray directed from p towards q
    $T_i(u) \leftarrow T_i(u) \cup \overline{p_k q}$
    $p \leftarrow q$ and $q \leftarrow p_k$ and $k \leftarrow k+1$
return $T_i(u)$

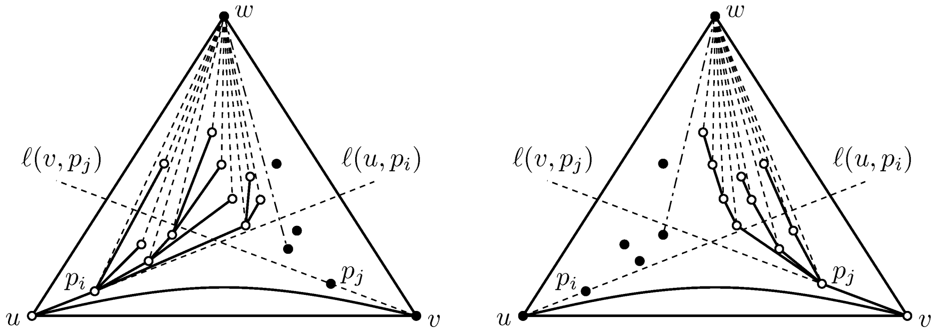

Figure 7: The tree $T_i(u)$ shown with white points in the left figure. The tree $T_j(v)$ shown with white points in the right figure, using the same set of points.

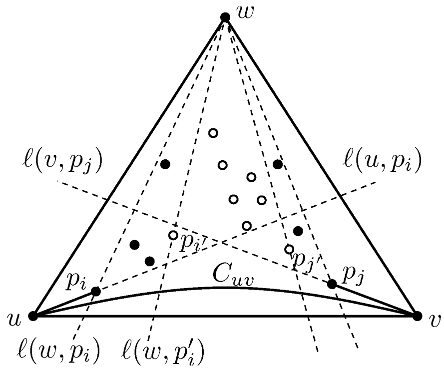

Figure 8: The special points $p_{i'}$ and $p_{j'}$, and the points from P' present in both trees are shown as white points.

Assume that we are interested in the minimum perimeter empty pseudo-triangle. From the root towards the leaves, we compute and store with each node (point) the Euclidean length of the path from that node to u. For each node, we then add the Euclidean distance from that node to w. By doing this, we have the length of the concave chain C_{uw} stored with each node, if that node is chosen as p_k.

A similar algorithm constructs the tree $T_j(v)$ consisting of edges of chains C_{vw} that end with $\overline{vp_j}$. It is shown at the right in Figure 7. Again we compute and store the chain lengths with each node, if that node is chosen as p_{k+1}.

Not all points of P' occur in both trees. Let j' be such that $p_{j'}$ is the point of P' with lowest index that lies below $\ell(u, p_i)$, see Figure 8. If no such point exists, we let $j' = j$. So $p_{j'}$ is the point where the tree construction of $T_i(u)$ halts. Similarly, let i' be such that $p_{i'}$ is the point of P' with highest index that lies below $\ell(v, p_j)$, or $i' = i$. Then we observe:

Lemma 3 *An empty pseudo-triangle exists with $\overline{wp_k}$ on C_{uw} and $\overline{wp_{k+1}}$ on C_{vw} if and only if $i' \leq k < k+1 \leq j'$.*

Our algorithm is therefore as follows:

Algorithm *Shortest-Pseudo-Triangle*
Input: A set P of n points inside a triangle $\triangle uvw$.
Output: An empty pseudo-triangle with minimum perimeter
1. Sort P counterclockwise in angular order around w
2. **for** all choices of p_i and p_j for which *Test-Pair* returns TRUE
3. Select the subset of points below $\ell(u, p_i)$ and below $\ell(v, p_j)$ and compute C_{uv}

Determine P' and compute $T_i(u)$ and $T_j(v)$ with *Rooted-Tree*
for $k \leftarrow i'$ **to** $j' - 1$
Locate p_k in $T_i(u)$ and p_{k+1} in $T_j(v)$ and add the stored values,
maintaining the minimum sum found so far
return the minimum perimeter empty pseudo-triangle

We analyze the running time of the algorithm *Shortest-Pseudo-Triangle.* The outer loop runs over $O(n^2)$ choices. *Test-Pair* trivially takes linear time. C_{uv} can be computed in linear time using the sorted order of points around w, ignoring those above $\ell(u, p_i)$ or above $\ell(v, p_j)$. The sorted sequence for P' can be extracted from the sorted sequence of P in linear time as well. The locations of p_k in $T_i(u)$ and p_{k+1} in $T_j(v)$ can be found in linear time overall, by traversal from the previous locations (every tree edge is traversed at most twice).

It remains to show that the trees $T_i(u)$ and $T_j(v)$ can be computed in linear time as well. The only time when adding the next point p_k to the tree may take more than constant time is when p_k is to the right of γ several consecutive times. However, each time, there is one node of the tree $T_{ij}(u)$ that will not be encountered again in the rest of the algorithm (namely q). Hence, the overall cost of these steps is still linear. We can compute all lengths in linear time using a simple tree traversal from the root u towards the leaves.

The adaptations needed to compute the maximum perimeter, minimum area, or maximum area empty pseudo-triangle are straightforward. If we are interested in the area measure, we store for each node the area of the convex hull of the path from that node to the root, instead of the length of that path.

Theorem 7 *Given n points inside a triangle, an empty pseudo-triangle with minimum or maximum perimeter, or minimum or maximum area can be computed in* $O(n^3)$ *time using linear space.*

4.2 Optimal empty pseudo-triangles in point sets

We now discuss the case where the three convex vertices are not given in advance, but can be chosen freely from a set P of n points in the plane. Recall that the maximum possible number of empty pseudo-triangles in this case is $\Theta(n^6)$. This leads to a straightforward adaptation of our algorithms to this case: Simply take all possible triples of points of P as the convex vertices of the pseudo-triangles and apply the appropriate algorithm. This obviously increases the running time by a factor of $O(n^3)$ in each case, without influencing the space requirement.

Theorem 8 *Given n points in the plane, an empty pseudo-triangle with minimum or maximum area, or minimum or maximum perimeter can be computed in* $O(n^6)$ *time using linear space.*

However, with a simple observation we can drastically reduce the running times for the two minimization problems, as we will show next. We start with minimizing the perimeter.

Lemma 4 *Given a set of points in the plane, any empty pseudo-triangle with the minimum perimeter is a triangle.*

Proof: Let $\mathcal{T}$ be a minimum perimeter pseudo-triangle with more than three vertices on its boundary. We can triangulate $\mathcal{T}$ and obtain empty triangles in $\mathcal{T}$ with smaller perimeter, a contradiction. $\square$

We can use a divide-and-conquer approach similar to the well-known closest pair algorithm to find the closest triple of points. We recursively divide the point set by a vertical line into two equal-size parts, until the subsets have size ≤ 5. The merge step is as follows: Let δ_1, δ_2 be the minimum perimeter length on each side of a dividing line ℓ, and let $\delta = \min\{\delta_1, \delta_2\}$. We need to check all points inside a strip of width $\delta/2$ to the left and right of ℓ. Following a similar argument as for finding the closest pair, we can show that each triple of points, not all on the same side of ℓ, must lie in a $\delta \times \delta$ rectangle, horizontally centered at ℓ, if they form a triangle with perimeter less than δ. Inside such a rectangle, there can be at most 16 points, hence, we only need to examine a constant number of possible triangles for each point inside the strip. This leads to a running time of $O(n \log n)$. We refer to the book by Cormen et al. [7] for further details of the algorithm.

Theorem 9 *Given n points in the plane, an empty pseudo-triangle with minimum perimeter can be computed in $O(n \log n)$ time using linear space.*

Next we discuss minimizing the area. We can use the same argument as above to prove that we are looking for a triangle.

Lemma 5 *Given a set of points in the plane, any empty pseudo-triangle with the minimum area is a triangle.*

Now we can use a result of Edelsbrunner and Guibas [10] and conclude:

Theorem 10 *Given n points in the plane, an empty pseudo-triangle with minimum area can be computed in $O(n^2)$ time using linear space.*

5 Conclusions

First, we have given tight bounds on the minimum and maximum number of empty pseudo-triangles that either must be star-shaped or may be arbitrary. The constructions and proofs are simple and elegant. An open question is whether pseudo-triangles that are 9-gons are necessary to have $\Omega(n^3)$ and $\Omega(n^6)$ empty pseudo-triangles in Theorems 1 and 4, or whether smaller complexity pseudo-triangles can also be used. Second, we have given algorithms to find optimal empty pseudo-triangles with respect to minimum and maximum perimeter or area. It would be interesting to improve the $O(n^6)$ time algorithms for the maximization problems.

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