Inscribing an axially symmetric polygon and other approximation algorithms for planar convex sets

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Abstract

Given a planar convex set C, we give sublinear approximation algorithms to determine approximations of the largest axially symmetric convex set S contained in P, and the smallest such set S' that contains P. More precisely, for any $\varepsilon > 0$, we find an axially symmetric convex polygon $Q \subset C$ with area $|Q| > (1-\varepsilon)|S|$ and we find an axially symmetric convex polygon Q' containing C with area $|Q'| < (1+\varepsilon)|S'|$. We assume that C is given in a data structure that allows to answer the following two types of query in time T_C: given a direction u, find an extreme point of C in direction u, and given a line ℓ, find $C \cap \ell$. For instance, if C is a convex n-gon and its vertices are given in a sorted array, then $T_C = O(\log n)$. Then we can find Q in time $O(T_C\varepsilon^{-1/2} + \varepsilon^{-3/2})$ and we can find Q' in time $O(T_C\varepsilon^{-1/2} + \varepsilon^{-3/2}\log(\varepsilon^{-1}))$. Using these techniques, we can also find approximations to the perimeter, area, diameter, width, smallest enclosing restangle and smallest enclosing circle of C in time $O(T_C\varepsilon^{-1/2})$.

1 Introduction

Some problems on convex polygons can be solved in sublinear time when the polygon P is given as an array of the n vertices in sorted order along the boundary of P. For instance, given a line ℓ, the two vertices of P that have tangents parallel to ℓ can be found in $O(\log n)$ time. The shortest line segment connecting two convex polygons can also be computed in $O(\log n)$ time [10]. Schwarzkopf et al. [20] showed how to compute a pair of rectangles approximating a given convex polygon in $O(\log^2 n)$ time. Kirkpatrick and Snoeyink [14] give a general framework that allows to answer several queries on a convex n-gon P in $O(\log n)$ time. Examples are the longest chord (or a chord of given length) parallel to a query line, or the largest homothet of a query triangle that fits inside P. Chazelle et al. [8] recently presented a different framework for obtaining sublinear time algorithms where the input is not given in sorted arrays, but in linked lists where random nodes can be accessed in constant time. It yields $O(\sqrt{n})$ time randomized algorithms for various problems, for instance for detecting intersections between convex polyhedra.

Other problems on convex polygons cannot be solved in sublinear time. For instance, determining the diameter or area of P takes $\Theta(n)$ time. In this paper we show that some of these problems can be solved in $O(\log n)$ time if an *approximate* solution is sufficient. We can, for instance, compute the diameter or the area of P up to a relative error of ε in time $O((\log n)/\sqrt{\varepsilon})$. In fact, we will give efficient algorithms for arbitrary compact convex sets in the plane. Our only assumption is that a convex set C is given in a data structure that allows to answer the following two types of queries in time T_C:

- given a query line ℓ, find $C \cap \ell$,

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- given a query direction u, find an extreme point in direction u.

For instance, if C is a convex n-gon given as an array of its vertices in counter-clockwise order, then we can answer these two types of queries in $O(\log n)$ time by binary search, so $T_C = O(\log n)$.

Our algorithms are based on an approximation of the input convex set C by a convex polygon whose size depends only on ε. This is not a new idea: our approximation is based on a constructive proof by Dudley from 1974 [9]. A paper by Agarwal et al. [1] uses this idea as well, some of these results have been improved recently by Chan [7]. Interestingly, these approximations can be computed in logarithmic time, a striking improvement compared, for instance, to the result by Lopez and Reisner [17]. They proposed an $O(n + (n-k)\log n)$ time algorithm for approximating a convex n-gon by an inscribed polygon with k vertices and relative approximation error $O(1/k^2)$. Our method achieves the same in time $O(k \log n)$ (or in time $O(n)$, independent of k).

In general, if it is possible to compute a certain property of a convex n-gon in time polynomial in n, and this property is *robust* with respect to approximation of the polygon, then our approximation technique immediately results in an approximation algorithm for an aritrary planar convex set C of running time roughly $O(\varepsilon^{-1/2} T_C + 1/\varepsilon^{O(1)})$. In the case where C is a convex n-gon given in an array, we obtain sublinear $O(\varepsilon^{-1/2} \log n + 1/\varepsilon^{O(1)})$ time algorithms.

We give some rather immediate applications of this technique, and then turn to our main result. We give logarithmic-time approximation algorithms to determine, for a given convex set C, approximations of the largest axially symmetric convex set contained in C, and the smallest such set that contains C.

There are a number of papers that study the best inner approximation of any convex set by a symmetric set; the distance to a symmetric set can be considered a measure of its symmetry [11]. Lower bounds for this distance are given by the Löwner-John ellipsoid [13]: any planar convex body C lies between two homothetic ellipses $E \subset C \subset 2E$ with homothety ratio at most 2. Since any ellipse is axially symmetric, and $\text{area}(E) = \frac{1}{4}\,\text{area}(2E) \geqslant \frac{1}{4}\,\text{area}(C)$, any convex planar set C contains an axially symmetric subset with at least $1/4$ of the area of C. The same lower bound of $1/4$ follows from the fact that any planar convex body lies between two homothetic rectangles with homothety ratio at most two [16, 20]. The lower bound can be raised to $2/3$ [15], a bound that is not known to be tight.

The largest centrally symmetric set contained in a convex shape C is the maximum intersection of C and a translate of $-C$. If C is a convex n-gon, this can be computed in $O(n \log n)$ time [5]. Approximation by axially symmetric sets is technically more demanding, as the largest axially symmetric set contained in C is the maximum intersection of C and a *rotated* and translated copy of C' (with C' a fixed axially reflected copy of C). We do not know of any exact algorithm to compute the maximum intersection of two convex polygons under translation and rotation (orientation-preserving rigid motions), indeed it is not clear that such an algorithm can exist within a reasonable model of computation.

A recent manuscript [4] proposes a practical algorithm to compute exactly the largest subset of a convex n-gon with an axial symmetry; but it requires to solve $\Theta(n^3)$ optimization problems for which no polynomial time algorithm is known. This is our motivation to give a fast approximation algorithm. We can find a $(1-\varepsilon)$-approximation in time $O(\varepsilon^{-1/2} T_C + \varepsilon^{-3/2})$.

The problem of outer approximation of a convex polygon by an axially symmetric polygon seems to have received less interest than inner approximation, perhaps because this is equivalent to the inner approximation problem if one drops the requirement that the axially symmetric polygon has to be convex. The results on approximation by homothetic pairs (ellipses or rectangles) cited above give again simple bounds: for each convex set C there is an axially symmetric set D containing C with $\text{area}(D) \leqslant 4\,\text{area}(C)$. The constant 4 can be reduced to $31/16$ [15], again this is probably not tight. We give an approximation algorithm for this problem with running time $O(\varepsilon^{-1/2} T_C + \varepsilon^{-3/2} \log(1/\varepsilon))$.

Both algorithms are based on three key ideas.

- First, as discussed before, we replace the input figure by a polygon with a number of vertices depending on ε only.

- Second, we discretize the set of directions and sample only directions in a discrete set. This works well as long as the polygon is not long and skinny. Fortunately we can show that for long and skinny polygons, the axis of an optimal symmetry must be very close to the diameter of the polygon, or must be nearly orthogonal to this diameter.
- Finally, we use an algorithm to compute the optimal solution for a given direction of the axis of symmetry. In the inscribed case, this is equivalent to finding the translation of C' that maximizes the area of $C \cap C'$. As mentioned before, this can be done in time $O(n \log n)$ [5]. In our case, it suffices to consider a one-dimensional set of translations, which permits a linear time solution [3]. This solution makes use of the area of cross-sections of a three-dimensional polytope and the Brunn-Minkowski Theorem. We do not know of a similarly efficient solution for the circumscribed case, so we give a plane sweep algorithm.

As mentioned before, the inscribed case is a special case of the problem of maximizing the overlap of two convex polygons C and C' under translation and rotation of C'. Surprisingly little is known about this problem. Alt et al. [2] made some initial progress on a similar problem, showing, for instance, how to construct, for a convex polygon P, the axis-parallel rectangle Q minimizing the symmetric difference of P and Q. Our solution does not generalize to this problem. It does not appear to be "robust" under approximation of C and C'. Furthermore, we do not know how to discretize the set of directions when C is fat while C' is long and skinny.

2 Notations

In this paper, all the convex sets we consider are compact and lie in the plane. So we will simply say *convex set* instead of planar compact convex set. Let C denote a convex set. We let $|C|$ denote the *area* of C, while $\operatorname{diam}(C)$ and $\operatorname{peri}(C)$ denote diameter and perimeter.

We denote by $\mathcal{U}$ the set of unit vectors in the plane. We identify a point M in the plane with the vector OM, where O is the origin. We denote by $\langle a, b\rangle$ the inner product of a and b. Let C be a convex set. The *directional width* of C in direction $u \in \mathcal{U}$ is the minimum width of a slab that contains C and is orthogonal to u. In other words, the directional width of C in direction u is:

$$\operatorname{dwidth}(u, C) = \max_{x \in C} \langle u, x\rangle - \min_{x \in C} \langle u, x\rangle.$$

The *width* of C is the minimum width along all the directions in $\mathcal{U}$, that is:

$$\operatorname{width}(C) = \min_{u \in \mathcal{U}} \left(\operatorname{dwidth}(u, C)\right).$$

We introduce another notion: for a convex set C, let $\operatorname{breadth}(C) := |P| / \operatorname{diam}(P)$. The name $\operatorname{breadth}(C)$ can be explained as follows: let pq be a diameter of C. There is then a rectangle R circumscribed to C with one side parallel to pq such that C touches all four sides of R. The sides of R have length $\operatorname{diam}(C)$ and w, and we have $\operatorname{diam}(C)w/2 = |R|/2 \leqslant |C| \leqslant |R| = \operatorname{diam}(C)w$. This implies $\operatorname{breadth}(C) \leqslant w \leqslant 2\operatorname{breadth}(C)$, so $\operatorname{breadth}(C)$ is an estimate for the directional width of C orthogonal to a diameter. (We use the word "breadth" instead of "width" to avoid confusion with the usual notion of width, which is explained in the previous paragraph.)

We assume that a convex set C is given in a data structure that allows to answer the following two types of queries in time T_C:

- given a line ℓ, find the line segment $C \cap \ell$.
- given a direction $u \in \mathcal{U}$, find a point x of C that is extreme along u. In other words,

$$\langle u, x\rangle = \max_{y \in C} \langle u, y\rangle.$$

For instance, if C is a convex n-gon whose vertices are given in a sorted array, we can answer these queries by binary search in time $O(\log n)$, so $T_C = O(\log n)$.

For two sets A and B such that $A \subset B$, the *Hausdorff-distance* between A and B is

$$d_H(A, B) := \max_{b \in B} \Big(\min_{a \in A} d(a, b) \Big)$$

where $d(a, b)$ is the Euclidean distance between a and b.

3 Preliminaries

We will make use of the following inequality [23, pg. 257, ex.7.17a].

Lemma 1 *For a convex set C we have* $\operatorname{peri}(C) \leqslant \pi \operatorname{diam}(C)$.

The following lemma bounds the increase in area when a convex set is enlarged.

Lemma 2 *Let C be a convex set, let $r > 0$, and let C' be the set of points at distance at most r from C (in other words, C' is the Minkowski sum of C and a disk of radius r). Then* $|C'| = |C| + r \operatorname{peri}(C) + \pi r^2$, $\operatorname{peri}(C') = \operatorname{peri}(C) + 2\pi r$, *and* $\operatorname{diam}(C') \leqslant \operatorname{diam}(C) + 2r$.

Proof. Assume first that C is a convex polygon. Then $C' \setminus C$ can be decomposed into rectangles of width r along each edge of C, and disk sectors at the vertices of C. The union of all the disk sectors is a disk of radius r, which implies the claim. For general C, approximate it by a sequence of polygons and take the limit. ◻

An alternate proof is suggested in Exercise 6 on page 47 of do Carmo [6]. A similar bound is the following "volume of tube" formula. Again one could prove this easily for convex polygons, and take the limit. The lemma also follows directly from the general volume-of-tube formula for smooth curves in any dimension by Hotelling [12] and Weyl [24].

Lemma 3 *Let C be a convex set, let $r > 0$, and let C' be the set of points at distance at most r from* the boundary of C *(in other words, C' is the Minkowski sum of* $\operatorname{bd} C$ *and a disk of radius r). Then* $|C'| \leqslant 2r \operatorname{peri}(C)$.

Finally, we bound the change in area incurred by a rotation around a point inside a convex polygon.

Lemma 4 *Let C be a convex set, and let C' be a copy of C, rotated by an angle δ around a point p in C. Then*

$$|C \cap C'| \geqslant |C| - \frac{\pi\delta}{2} \operatorname{diam}(C)^2.$$

Proof. We denote by D the symmetric difference between C and C', in other words $D = (C \cup C') \setminus (C \cap C')$. We denote by C_m the copy of C rotated by an angle $\delta/2$ around p. Let T_m denote the set of points that are at distance at most $\delta \operatorname{diam}(C)/2$ from the boundary of C_m. Note that any point q in D is obtained from a point of C_m by a rotation of center p and angle at most $\delta/2$ in absolute value. Since the distance pq is at most $\operatorname{diam}(C)$, it follows that $q \in T_m$. Thus $D \subset T_m$. By Lemma 3, the area of T_m is at most $\delta \operatorname{diam}(C) \operatorname{peri}(C_m)$. Since $\operatorname{peri}(C_m) = \operatorname{peri}(C)$ and, by Lemma 1, $\operatorname{peri}(C) \leqslant \pi \operatorname{diam}(C)$ we obtain that $|T_m| \leqslant \pi\delta \operatorname{diam}(C)^2$. Since $D \subset T_m$, it implies that $|D| \leqslant \pi\delta \operatorname{diam}(C)^2$. The result follows from $|D| = |C| - |C \cap C'| + |C'| - |C \cap C'| = 2(|C| - |C \cap C'|)$. ◻

4 Approximating a convex set

A key component of our proofs is a polygon approximation whose size depends only on ε. In particular, we will show that the framework of Agarwal et al. [1] can be implemented efficiently in the case of planar convex sets. We start with a lemma.

Lemma 5 *Given a convex set C, we can find in $O(T_C)$ time a rectangle R with sides a, b containing C such that C touches all four sides of R and such* $\operatorname{diam}(C)/\sqrt{2} \leqslant a \leqslant \operatorname{diam}(C)$, $\operatorname{breadth}(C) \leqslant b \leqslant 4\operatorname{breadth}(C)$, *and* $|R|/(2\sqrt{2}) \leqslant |C| \leqslant |R|$.

Proof. We can determine a point of C that is extreme in a given direction in time T_C. By doing this four times, we can find the axis-parallel bounding box R' of C. Let $a' \geqslant b' > 0$ be its sides, and pick the vertices p, q of C touching the shorter sides of R'. Then $a' \leqslant d(p,q) \leqslant \operatorname{diam}(C) \leqslant \operatorname{diam}(R) \leqslant \sqrt{2}a'$. We now compute the smallest rectangle R containing C with a side parallel to pq. The side parallel to pq has length $a \geqslant d(p,q) \geqslant a' \geqslant \operatorname{diam}(C)/\sqrt{2}$. Since C contains two triangles with common base pq and total height b, we have $ab = |R| \geqslant |C| \geqslant d(p,q)b/2 \geqslant ab/(2\sqrt{2}) = |R|/(2\sqrt{2})$. Finally, we have $b \geqslant |C|/a \geqslant |C|/\operatorname{diam}(C) = \operatorname{breadth}(C)$, and $b \leqslant 2\sqrt{2}|C|/a \leqslant 2\sqrt{2}|C|/(\operatorname{diam}(C)/\sqrt{2}) = 4\operatorname{breadth}(C)$. ⊔⊓

Following Agarwal et al. [1], we say that a convex set $C' \subset C$ is an ε-kernel of C if and only if

$$\forall u \in \mathcal{U},\ (1-\varepsilon)\operatorname{dwidth}(u, C) \leqslant \operatorname{dwidth}(u, C').$$

We give an efficient algorithm to compute a low-complexity ε-kernel of a convex set C. It is based on Dudley's constructive proof [9]. Note that the running time of the linear-time version of the algorithm has no dependence on ε at all.

Lemma 6 *Given a planar convex set C and $\varepsilon > 0$, one can construct in time $O(T_C/\sqrt{\varepsilon})$ two convex polygons C_ε and C'_ε with $O(1/\sqrt{\varepsilon})$ vertices such that $C_\varepsilon \subset C \subset C'_\varepsilon$ and $|C'_\varepsilon \setminus C_\varepsilon| \leqslant \varepsilon|C|$. In addition, C_ε is an ε-kernel of C, and C is an ε-kernel of C'_ε. If C is a convex n-gon, then we can compute C_ε and C'_ε in time $O(n)$.*

Proof. We start by computing a rectangle R as in Lemma 5, and apply a transformation that maps R to the unit square. Ratios of area and directional width are invariant under affine transformations. In the following, we will therefore assume that C is inscribed in a unit square R.

First we prove a lower bound on $\operatorname{width}(C)$. By Lemma 5, we have $|C| \geqslant 1/2\sqrt{2}$. Let u_0 be a direction such that $\operatorname{width}(C) = \operatorname{dwidth}(u_0, C)$ and u_1 be a direction orthogonal to u_0. Clearly $\operatorname{dwidth}(u_1, C) \leqslant \operatorname{diam}(R) = \sqrt{2}$. Therefore

$$\frac{1}{2\sqrt{2}}|C| \leqslant \operatorname{dwidth}(u_1, C)\operatorname{dwidth}(u_0, C) \leqslant \sqrt{2}\operatorname{width}(C).$$

so $\operatorname{width}(C) \geqslant 1/4$.

We now discuss the linear-time algorithm for the case that C is a convex n-gon. We go once around C, starting at an arbitrary vertex, and select edges of C as we go. We always choose the first edge. Let $e = ww'$ be the most recently chosen edge, let $e' = vv'$ be the next candidate edge, and let $e'' = v'v''$ be the edge following e'. We choose e' if

- the distance $d(w', v') > \sqrt{\varepsilon}/3$, or
- the outer normals of e and e'' make an angle larger than $\sqrt{\varepsilon}/3$.

We observe that the number of edges selected is $O(1/\sqrt{\varepsilon})$. Remember that C is inscribed in a unit square, so by Lemma 1, only $O(1/\sqrt{\varepsilon})$ edges can be chosen according to the first rule. The total change of the outer normal angles is $2\pi = O(1)$, so only $O(1/\sqrt{\varepsilon})$ edges can be chosen according to the second rule.

Let C_ε be the convex hull of the selected segments, and let C'_ε be the polygon obtained by extending the selected edges until they form a convex polygon. Then $C_\varepsilon \subset C \subset C'_\varepsilon$.

The difference $C'_\varepsilon \setminus C_\varepsilon$ consists of $O(1/\sqrt{\varepsilon})$ triangles $ww'v$, where w and w' are vertices of C, and v is the intersection of the lines supporting two consecutive selected edges. The distance $d(w, w') \leqslant \sqrt{\varepsilon}/3$, the lines wv and $w'v$ make an angle of at most $\sqrt{\varepsilon}/3$, and so the angle $\angle wvw' \geqslant \pi - \sqrt{\varepsilon}/3$. Together this implies that the height of the triangle is at most $\varepsilon/9$, and so $d_H(C_\varepsilon, C'_\varepsilon) \leqslant \varepsilon/9$. In particular, since $C_\varepsilon \subset C \subset C'_\varepsilon$, it follows that $d_H(C_\varepsilon, C) \leqslant \varepsilon/9$ and

$d_H(C, C'_\varepsilon) \leqslant \varepsilon/9$. So for all directions $u \in \mathcal{U}$, we have $\mathrm{dwidth}(u, C) - 2\varepsilon/9 \leqslant \mathrm{dwidth}(u, C_\varepsilon)$, and $\mathrm{dwidth}(u, C'_\varepsilon) - 2\varepsilon/9 \leqslant \mathrm{dwidth}(u, C)$. Remember that $\mathrm{width}(C) > 1/4$, so $\mathrm{dwidth}(u, C) > 1/4$. Therefore $(1-\varepsilon)\,\mathrm{dwidth}(u, C) \leqslant \mathrm{dwidth}(u, C_\varepsilon)$, and C_ε is an ε-kernel for C. Since $\mathrm{dwidth}(u, C'_\varepsilon) \geqslant \mathrm{dwidth}(u, C) > 1/4$, we also have $(1-\varepsilon)\,\mathrm{dwidth}(u, C'_\varepsilon) \leqslant \mathrm{dwidth}(u, C)$, and C is an ε-kernel for C'_ε.

We have just observed that $C'_\varepsilon \setminus C_\varepsilon$ consists of triangles with height $\varepsilon/9$ and such that the sum of the length of their bases is at most $\mathrm{peri}\, C_\varepsilon$. Since C_ε is contained in a unit square, its perimeter is at most 4. So $|C'_\varepsilon \setminus C_\varepsilon| \leqslant 2\varepsilon/9 < \varepsilon/2\sqrt{2} \leqslant \varepsilon|C|$.

Let now C be a (not necessarily polygonal) convex set. We will show how to compute C_ε and C'_ε in time $O(T_C/\sqrt{\varepsilon})$. We will select a sequence of points $p_1, p_2, \ldots, p_s$ on the boundary of C such that the following holds (let $p_0 := p_s$):

- $p_1, p_2, \ldots, p_s$ is sorted in counter-clockwise order along the boundary of C,
- $s = O(1/\sqrt{\varepsilon})$,
- for $i = 1, \ldots, s$, there are tangents to C in p_{i-1} and p_i that make an angle of at most $\sqrt{\varepsilon}/3$, and
- for $i = 1, \ldots, s$, the distance between p_{i-1} and p_i is at most $\sqrt{\varepsilon}/3$.

Let C_ε be the convex hull of $p_1, \ldots, p_s$, and let C'_ε be the polygon formed by the at most $2s$ tangents to C in $p_1, \ldots, p_s$. Then $C_\varepsilon \subset C \subset C'_\varepsilon$, and $C'_\varepsilon \setminus C_\varepsilon$ consists of $s = O(1/\sqrt{\varepsilon})$ triangles $p_{i-1}p_i v$. The approximation bounds now follow as in the polygon case.

To compute C_ε, we first select the boundary points of C that are extreme in a set of $3/\sqrt{\varepsilon}$ equally spaced directions. We then consider a set of equally spaced horizontal and vertical lines at distance $\sqrt{\varepsilon/18}$, and select the points of intersection between the boundary of C and these lines. This takes $O(1/\sqrt{\varepsilon})$ queries on the convex set C, and results in a sequence of points as required above. We obtain C_ε as the convex hull of the selected point sequence.

The outer approximation C'_ε takes a little more work: the difficulty is that we do not know the tangents in the boundary points obtained by line intersection queries. We therefore first compute the inner approximation $C_{\varepsilon/2}$ (that is, with $\varepsilon' = \varepsilon/2$). For each edge $p_{i-1}p_i$ of $C_{\varepsilon/2}$, let u_i be the outer normal of $p_{i-1}p_i$. We compute the point $q_i \in C$ extreme in direction u_i. The sequence $q_1, q_2, \ldots$ now fulfills the requirements above, and we easily obtain C'_ε given the points and the tangent directions. ◻

Following Agarwal et al. [1], we define *faithful measures* for convex sets. A function μ is a faithful measure if $\mu(C) \geqslant 0$ for any convex set C and if there exists a constant $c > 0$ such that, for any ε-kernel C_ε of C, we have $(1 - c\varepsilon)\mu(C) \leqslant \mu(C_\varepsilon) \leqslant \mu(C)$. We list a few measures that were shown to be faithful by Agarwal et al. [1].

Lemma 7 ([1], Section 6.1) *The following measures $\mu(C)$ are faithful:*

(a) diameter $\mathrm{diam}(C)$,

(b) width $\mathrm{width}(C)$,

(c) area $|C|$,

(d) perimeter $\mathrm{peri}(C)$,

(e) the radius of the smallest enclosing disk of C,

(f) the area of the smallest enclosing rectangle of C.

5 Simple applications

We give a few simple applications of our approximation technique to optimization problems for convex sets. Following Agarwal et al. [1], we first compute a kernel of the convex input set, and then we apply known algorithms on the kernel. For a number of problems, this provides an approximate solution to the optimization problem on the original convex set.

Theorem 8 *Given a planar convex set C, we can compute a $(1-\varepsilon)$-approximation of its area, diameter, perimeter and width in time $O(T_C/\sqrt{\varepsilon})$. In particular, if C is a convex n-gon and its vertices are given in a sorted array or a balanced binary search tree, then we can compute these approximations in time $O(\log n/\sqrt{\varepsilon})$. We can also compute $(1+\varepsilon)$-approximations of the smallest area enclosing rectangle and the smallest enclosing disk of C within the same time bounds.*

Proof. The area and perimeter of a convex n-gon can be easily computed in $O(n)$ time. Its diameter, width and smallest area enclosing rectangle can also be computed in $O(n)$ time using, for instance, the rotating callipers technique of Toussaint [21]. The smallest enclosing disk can also be found in $O(n)$ time [18].

By Lemma 7, all these measures are faithful, and so there is a constant $c > 0$ such that an ε/c-kernel $C_{\varepsilon/c}$ for C provides $(1-\varepsilon)$-approximations of diameter, width, area, and perimeter of C. We can compute $C_{\varepsilon/c}$ in time $O(T_C/\sqrt{\varepsilon})$. It has $O(1/\sqrt{\varepsilon})$ vertices, and so we can compute its diameter, width, area, and perimeter within the same time bound.

For the smallest enclosing rectangle and disk, we use the outer approximation $C'_{\varepsilon/c}$ for a suitable $c > 0$ instead. Since C is an ε/c-kernel of $C'_{\varepsilon/c}$, the smallest enclosing rectangle and disk of $C'_{\varepsilon/c}$ are $(1+\varepsilon)$-approximations of the smallest enclosing rectangle and disk of C. Again, they can be computed in time $O(T_C/\sqrt{\varepsilon})$. ◻

6 The largest axially symmetric inscribed set

In the following we denote by $\mathrm{refl}(\cdot,\ell)$ the reflection at line ℓ, so that $\mathrm{refl}(C,\ell)$ is the reflected image of C under reflection at ℓ. Let C be a convex set in the plane and let ℓ be a line. The set $C \cap \mathrm{refl}(C,\ell)$, if it is not empty, is an axially symmetric convex subset of C, the largest axially symmetric subset with reflection axis ℓ. Our goal is to find, for a convex set C, a line $\ell^{\mathrm{opt}}(C)$ that maximizes the area of this set:

$$\left|C \cap \mathrm{refl}(C,\ell^{\mathrm{opt}}(C))\right| = \max_{\ell\subset\mathbb{R}^2}\left|C \cap \mathrm{refl}(C,\ell)\right|.$$

As we discussed in the introduction, Lassak proved the following lower bound [15]:

Lemma 9

$$\left|C \cap \mathrm{refl}(C,\ell^{\mathrm{opt}}(C))\right| \geqslant \frac{2}{3}|C|$$

Our main result shows that at least an ε-approximation ℓ_ε with

$$(1-\varepsilon)\left|C \cap \mathrm{refl}(C,\ell^{\mathrm{opt}}(C))\right| < \left|C \cap \mathrm{refl}(C,\ell_\varepsilon)\right|$$

can be found fast.

If the direction of ℓ is known, we can compute the optimal line using the following lemma.

Lemma 10 *Given a convex n-gon P and a line ℓ, one can find in $O(n)$ time the line ℓ' parallel to ℓ that maximizes $\left|P \cap \mathrm{refl}(P,\ell')\right|$.*

Proof. Let $Q := \mathrm{refl}(P, \ell)$, and let t be a vector orthogonal to ℓ. For any line ℓ' parallel to ℓ, $\mathrm{refl}(P, \ell')$ is a translation of Q by a multiple of t, and so the problem is equivalent to finding $\lambda \in \mathbb{R}$ such that $|P \cap (Q + \lambda t)|$ is maximized. A linear-time algorithm to solve this problem has been given by Avis et al. [3]. ◻

We will apply this algorithm to a set of $O(1/\varepsilon)$ directions. The following two lemmas show how to find this set of directions.

Lemma 11 *Let ℓ and ℓ' be two lines intersecting in a point p with an angle δ, and let C be a convex set. If $p \in C \cap \mathrm{refl}(C, \ell)$, then*

$$|C \cap \mathrm{refl}(C, \ell')| \geqslant |C \cap \mathrm{refl}(C, \ell)| - \pi\delta\,\mathrm{diam}(C)^2.$$

Proof. The concatenation of the reflection at ℓ and the reflection at ℓ' is a rotation around p by the angle 2δ. Let $Q := C \cap \mathrm{refl}(C, \ell)$. Since Q is symmetric with respect to ℓ, the set $\mathrm{refl}(Q, \ell') = \mathrm{refl}(\mathrm{refl}(Q, \ell), \ell')$ is a copy of Q rotated around p by 2δ. Since $p \in Q$, Lemma 4 implies that

$$|Q \cap \mathrm{refl}(Q, \ell')| \geqslant |Q| - \pi\delta\,\mathrm{diam}(Q)^2.$$

Since $Q \subset C$, we have

$$C \cap \mathrm{refl}(C, \ell') \supset Q \cap \mathrm{refl}(Q, \ell'),$$

and by the above that implies

$$|C \cap \mathrm{refl}(C, \ell')| \geqslant |Q| - \pi\delta\,\mathrm{diam}(Q)^2 \geqslant |C \cap \mathrm{refl}(C, \ell)| - \pi\delta\,\mathrm{diam}(C)^2.$$ ◻

The occurrence of $\mathrm{diam}(C)^2$ instead of $|C|$ is a problem. In the following lemma, we will need to give special consideration to the case where the set C is long and skinny, that is, when $\mathrm{diam}(C)^2$ is much larger than $|C|$. Intuitively, when C is fat we will just sample the space of directions uniformly. When C is long and skinny, we will sample more densely, but we will only sample near the two axes of symmetry of a bounding rectangle R that is parallel to a diametral segment ab (see Fig. 1).

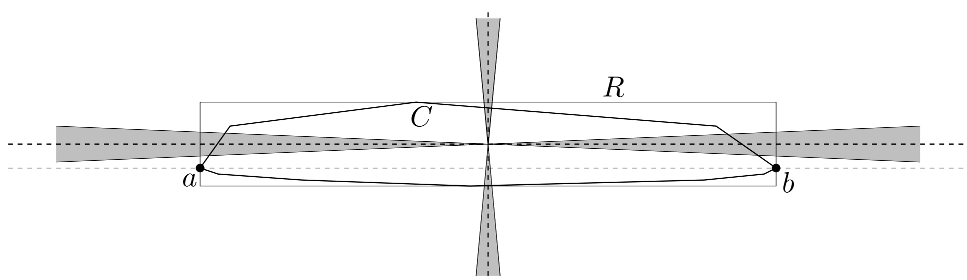

Figure 1: Case where C is long and skinny. We sample directions from the shaded area.

Lemma 12 *Given a convex n-gon P and $\varepsilon > 0$, one can construct in time $O(n + 1/\varepsilon)$ a set D_ε of $O(1/\varepsilon)$ directions such that*

$$(1 - \tfrac{1}{2}\varepsilon)\Big|P \cap \mathrm{refl}(P, \ell^{\mathrm{opt}}(P))\Big| \leqslant \max\big\{\big|P \cap \mathrm{refl}(P, \ell)\big| : \ \ell \textit{ has a direction from } D_\varepsilon \ \big\}.$$

Proof. By Lemma 11 and Lemma 9 it is sufficient to choose the set D_ε such that it contains a line that makes an angle δ of at most $\varepsilon|P|/(3\pi\,\mathrm{diam}(P)^2)$ with ℓ^{opt}.

We start by computing, in time $O(n)$, a diameter pq of P, and the area $|P|$. We then distinguish two cases.

If $\operatorname{diam}(P)^2 \leqslant 20|P|$, then we generate D_ε by sampling the direction space uniformly, choosing multiples of $\varepsilon/30\pi$. Since $\varepsilon/60\pi \leqslant \varepsilon|P|/3\pi \operatorname{diam}(P)^2$, this is sufficient.

If, on the other hand, $\operatorname{diam}(P)^2 > 20|P|$, then we sample uniformly the directions within $3\pi|P|/2 \operatorname{diam}(P)^2$ of the direction of the diameter ab, choosing multiples of $2\varepsilon|P|/3\pi \operatorname{diam}(P)^2$. We do the same around the direction that is orthogonal to ab. To show that this is sufficient we have to demonstrate that $\ell^{\text{opt}}(P)$ does not make an angle larger than $3\pi|P|/2 \operatorname{diam}(P)^2$ with the direction of the diameter or with the direction that is orthogonal to the diameter.

As in the argument at the beginning of Section 3, let R be the rectangle circumscribed to P with a side parallel to pq. The longer side of R has length $\operatorname{diam}(P)$, and P touches all four sides of R. This implies that $|R| \leqslant 2|P|$, and so its width w is at most $2|P|/\operatorname{diam}(P) = 2 \operatorname{breadth}(P)$. It follows that P lies in an infinite strip of width at most $2 \operatorname{breadth}(P)$. Let $\gamma \in [0, \pi/2]$ be the angle made by the lines $\ell^{\text{opt}}(P)$ and pq. The set $\operatorname{refl}(P, \ell^{\text{opt}})$ is contained in a congruent strip, intersecting the strip of P at an angle 2γ. The set $P \cap \operatorname{refl}(P, \ell^{\text{opt}})$ is contained in the intersection of the two strips, which has area $4 \operatorname{breadth}(P)^2/\sin 2\gamma$. By Lemma 9, we know that $|P \cap \operatorname{refl}(P, \ell^{\text{opt}}(P))| \geqslant 2|P|/3$, so the angle γ must satisfy

$$\frac{4 \operatorname{breadth}(P)^2}{\sin 2\gamma} \geqslant \frac{2}{3}|P|,$$

thus $\sin 2\gamma \leqslant 6|P|/\operatorname{diam}(P)^2$. It means that we are in one of the following two cases: $\gamma \leqslant 3\pi|P|/2 \operatorname{diam}(P)^2$ or $\pi/2 - \gamma \leqslant 3\pi|P|/2 \operatorname{diam}(P)^2$. ◻

We can now state our main theorem.

Theorem 13 *Let C be a planar convex set. Given $\varepsilon > 0$, we can find a set $Q \subset C$ with axial symmetry and*

$$\operatorname{area}(Q) \geqslant (1-\varepsilon) \max\left\{ \operatorname{area}(Q^*) \;\middle|\; Q^* \subset C \text{ and } Q^* \text{ axially symmetric} \right\}$$

in time $O(T_C \varepsilon^{-1/2} + \varepsilon^{-3/2})$.

Proof. We first construct the outer approximating polygon C'_{ε_1} of Lemma 6 with $\varepsilon_1 := \varepsilon/6$, obtain for this polygon a set of $O(1/\varepsilon)$ directions from Lemma 12, and determine for each of them the optimal line by Lemma 10.

It takes time $O(T_C \varepsilon^{-1/2})$ to construct C'_{ε_1}, time $O(1/\sqrt{\varepsilon} + 1/\varepsilon) = O(1/\varepsilon)$ to construct D_ε, and for each of the $O(1/\varepsilon)$ directions it takes time $O(1/\sqrt{\varepsilon})$ to find the optimal line of that direction. Together this is the claimed complexity of $O(\varepsilon^{-1/2} T_C + \varepsilon^{-3/2})$.

It remains to show that the line ℓ_ε with the largest intersection gives an approximation as claimed.

$$\begin{aligned}
(1 - \tfrac{1}{2}\varepsilon)\left| C \cap \operatorname{refl}(C, \ell^{\text{opt}}(C)) \right| &\leqslant (1 - \tfrac{1}{2}\varepsilon)\left| C'_{\varepsilon_1} \cap \operatorname{refl}(C'_{\varepsilon_1}, \ell^{\text{opt}}(C)) \right| \\
&\leqslant (1 - \tfrac{1}{2}\varepsilon)\left| C'_{\varepsilon_1} \cap \operatorname{refl}(C'_{\varepsilon_1}, \ell^{\text{opt}}(C'_{\varepsilon_1})) \right| \\
&\leqslant \left| C'_{\varepsilon_1} \cap \operatorname{refl}(C'_{\varepsilon_1}, \ell_\varepsilon) \right| \\
&\leqslant \left| C \cap \operatorname{refl}(C, \ell_\varepsilon) \right| + 2\left| C'_{\varepsilon_1} \setminus C \right| \\
&\leqslant \left| C \cap \operatorname{refl}(C, \ell_\varepsilon) \right| + \tfrac{1}{3}\varepsilon |C| \\
&\leqslant \left| C \cap \operatorname{refl}(C, \ell_\varepsilon) \right| + \tfrac{1}{2}\varepsilon \left| C \cap \operatorname{refl}(C, \ell^{\text{opt}}(C)) \right|.
\end{aligned}$$

In the last inequality we used Lemma 9. It follows that

$$(1-\varepsilon)\left| C \cap \operatorname{refl}(C, \ell^{\text{opt}}(C)) \right| \leqslant \left| C \cap \operatorname{refl}(C, \ell_\varepsilon) \right|,$$

which completes the proof. ◻

7 The smallest axially symmetric circumscribed convex set

Consider again a convex set C in the plane and a line ℓ. The set $\operatorname{conv}\big(C \cup \operatorname{refl}(C,l)\big)$ is an axially symmetric convex superset of C, the smallest axially symmetric convex superset with reflection axis ℓ. We want to find a line $\ell^{\mathrm{opt}}(C)$ that minimizes the area of this set:

$$\Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell^{\mathrm{opt}}(C))\big)\Big| = \min_{\ell \subset \mathbb{R}^2} \Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C,\ell)\big)\Big|.$$

As we discussed in the introduction, Lassak proved the following upper bound [15] (in fact, he proved a slightly stronger bound):

Lemma 14

$$\Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell^{\mathrm{opt}}(C))\big)\Big| \leqslant 2|C|$$

The main result of this section shows that at least an ε-approximation ℓ_ε with

$$(1+\varepsilon)\Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell^{\mathrm{opt}}(C))\big)\Big| > \Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell_\varepsilon)\big)\Big|$$

can be found fast. As in the previous section, we make use of a subroutine to find the optimal solution for a given direction of ℓ:

Lemma 15 *Given a convex n-gon P and a line ℓ, one can find in $O(n \log n)$ time the line ℓ' parallel to ℓ that minimizes* $\Big|\operatorname{conv}\big(P \cup \operatorname{refl}(P, \ell')\big)\Big|$.

We will prove this lemma after finishing the proof of the main theorem.

Again, we apply the subroutine to a set of directions that we obtain using the following two lemmas.

Lemma 16 *Let ℓ and ℓ' be two lines intersecting in a point p with an angle δ, and let C be a convex set. If $p \in C$, then*

$$|\operatorname{conv}(C \cup \operatorname{refl}(C,\ell'))| \leqslant |\operatorname{conv}(C \cup \operatorname{refl}(C,\ell))| + 4\pi(1+\pi/2)\delta \operatorname{diam}(C)^2.$$

Proof. Let $Q := \operatorname{conv}(C \cup \operatorname{refl}(C,\ell))$ and $Q' := \operatorname{conv}(C \cup \operatorname{refl}(C,\ell'))$. As in Lemma 4, we argue that any point of $\operatorname{refl}(C,\ell')$ has distance at most $2\delta \operatorname{diam}(C)$ from some point of $\operatorname{refl}(C,\ell)$. This implies that Q' is contained in the Minkowski-sum of Q with a disk of radius $2\delta \operatorname{diam}(C)$. By Lemma 2, this implies

$$|Q'| \leqslant |Q| + 2\delta \operatorname{diam}(C) \operatorname{peri}(Q) + \pi(2\delta \operatorname{diam}(C))^2.$$

Since $p \in C$, we have $C \cap \operatorname{refl}(C,\ell) \neq \emptyset$, and so $\operatorname{diam}(Q) \leqslant 2 \operatorname{diam}(C)$. This implies $\operatorname{peri}(Q) \leqslant 2\pi \operatorname{diam}(C)$, and we obtain

$$|Q'| \leqslant |Q| + 4\pi(\delta + \delta^2) \operatorname{diam}(C)^2 \leqslant |Q| + 4\pi(1+\pi/2)\delta \operatorname{diam}(C)^2.$$

□

Lemma 17 *Given a convex n-gon P and $\varepsilon > 0$, one can construct in time $O(n + 1/\varepsilon)$ a set D_ε of $O(1/\varepsilon)$ directions such that*

$$(1+\tfrac{1}{3}\varepsilon)\Big|\operatorname{conv}\big(P \cup \operatorname{refl}(P, \ell^{\mathrm{opt}}(P))\big)\Big| \geqslant \min\Big\{\big|\operatorname{conv}\big(P \cup \operatorname{refl}(P,\ell)\big)\big| : \ \ell \text{ has a direction from } D_\varepsilon\Big\}.$$

Proof. By Lemma 16 it is sufficient to choose the set D_ε such that it contains a line that makes an angle δ of at most $\varepsilon|C|/12\pi(1+\pi/2)\operatorname{diam}(C)^2$ with ℓ^{opt}. Again we distinguish two cases, depending on the ratio $|C|/\operatorname{diam}(C)^2$.

If $\operatorname{diam}(C)^2 \leqslant 10|C|$, then we generate D_ε by sampling the direction space uniformly, choosing multiples of $\varepsilon/1000$.

If $\operatorname{diam}(C)^2 > 10|C|$, then we generate D_ε as follows. We sample uniformly the directions within $\pi|C|/\operatorname{diam}(C)^2$ of the direction of the diameter pq, choosing multiples of $\varepsilon|C|/100\operatorname{diam}(C)^2$. We also sample in the same way around the direction orthogonal to pq (see Fig. 1).

To show that this is sufficient, notice that if $\ell^{\text{opt}}(C)$ intersects pq at an angle $\gamma \in [0, \pi/2]$, then $\operatorname{conv}(C \cup \operatorname{refl}(C, \ell^{\text{opt}}(C)))$ contains the diametral pair pq together with its reflected version $p'q'$, and pq makes an angle 2γ with $p'q'$. Therefore

$$2|C| \geqslant \Big|\operatorname{conv}(C \cup \operatorname{refl}(C, \ell^{\text{opt}}(C)))\Big| \geqslant \Big|\operatorname{conv}(\{p, q, p', q'\})\Big| \geqslant \frac{1}{2}\operatorname{diam}(C)^2 \sin 2\gamma.$$

Here we used Lemma 14. It follows that $\sin 2\gamma \leqslant 4|C|/\operatorname{diam}(C)^2$, and so we are in one of the following two cases: $\gamma \leqslant \pi|C|/\operatorname{diam}(C)^2$ or $\pi/2 - \gamma \leqslant \pi|C|/\operatorname{diam}(C)^2$. ◻

In order to replace the given input figure by a kernel, we need to show that the area of the smallest axially symmetric convex set containing C is a faithful measure. We use the following lemma.

Lemma 18 *Let ℓ be a line in the plane. Then the following measure is faithful:*

$$\mu(C) := |\operatorname{conv}(C \cup \operatorname{refl}(C, \ell))|.$$

Proof. Let C_ε be an ε-kernel of C. It is easy to see that then $\operatorname{conv}\big(C_\varepsilon \cup \operatorname{refl}(C_\varepsilon, \ell)\big)$ is an ε-kernel of $\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell)\big)$. The claim now follows from Lemma 7 (c). ◻

We can now prove the main result of this section.

Theorem 19 *Let C be a convex set in the plane. Given $\varepsilon > 0$, we can find a convex set $Q \supset C$ with axial symmetry and*

$$\operatorname{area}(Q) < (1+\varepsilon)\min\Big\{\operatorname{area}(Q^*) \;\Big|\; Q^* \supset C \text{ and } Q^* \text{ convex and axially symmetric}\Big\}$$

in time $O(\varepsilon^{-1/2}T_C + \varepsilon^{-3/2}\log(\varepsilon^{-1}))$.

Proof. We first construct the inner approximating polygon C_{ε_1} of Lemma 6 with $\varepsilon_1 = \varepsilon/c$ for a suitable constant $c > 0$, obtain for this polygon a set of $O(1/\varepsilon)$ directions from Lemma 17, and determine for each of them the optimal line by Lemma 15. The procedure takes time $O(\varepsilon^{-1}T_C + \varepsilon^{-3/2}\log(\varepsilon^{-1}))$ in total.

The constant $c > 0$ is chosen such that

$$(1-\varepsilon/3)|\operatorname{conv}(C \cup \operatorname{refl}(C, \ell))| \leqslant |\operatorname{conv}(C_{\varepsilon_1} \cup \operatorname{refl}(C_{\varepsilon_1}, \ell))|$$

for any line ℓ. This is possible by Lemma 18.

It remains to show that the line ℓ_ε minimizing $|\operatorname{conv}(C_{\varepsilon_1} \cup \operatorname{refl}(C_{\varepsilon_1}, \ell_\varepsilon))|$ among all lines with directions from D_ε is the required approximation.

$$\begin{aligned}(1+\tfrac{1}{3}\varepsilon)\Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell^{\text{opt}}(C))\big)\Big| &\geqslant (1+\tfrac{1}{3}\varepsilon)\Big|\operatorname{conv}\big(C_{\varepsilon_1} \cup \operatorname{refl}(C_{\varepsilon_1}, \ell^{\text{opt}}(C))\big)\Big| \\ &\geqslant (1+\tfrac{1}{3}\varepsilon)\Big|\operatorname{conv}\big(C_{\varepsilon_1} \cup \operatorname{refl}(C_{\varepsilon_1}, \ell^{\text{opt}}(C_{\varepsilon_1}))\big)\Big| \\ &\geqslant \Big|\operatorname{conv}\big(C_{\varepsilon_1} \cup \operatorname{refl}(C_{\varepsilon_1}, \ell_\varepsilon)\big)\Big| \\ &\geqslant (1-\tfrac{1}{3}\varepsilon)\Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell_\varepsilon)\big)\Big|\end{aligned}$$

For ε small enough, it follows that

$$(1+\varepsilon)\Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell^{\text{opt}}(C))\big)\Big| > \Big|\operatorname{conv}\big(C \cup \operatorname{refl}(C, \ell_\varepsilon)\big)\Big|.$$

◻

It remains to give the algorithm for Lemma 15. The approach used by Avis et al. [3], using cross-sections of a three-dimensional polytope and the Brunn-Minkowski Theorem, does not seem to apply here—in fact we do not know whether the area function has a single local minimum. We give a plane-sweep algorithm that runs in time $O(n \log n)$.

We use a coordinate system such that ℓ is the x-axis. For any $t \in \mathbb{R}$, we denote by P_t the polygon obtained from P by a reflexion at the line $y = t$. We denote by H_t the part of $\text{conv}(P \cup P_t)$ that lies on or above the line $y = t$ (see Fig. 2). We want to minimize the area of $\text{conv}(P \cup P_t)$

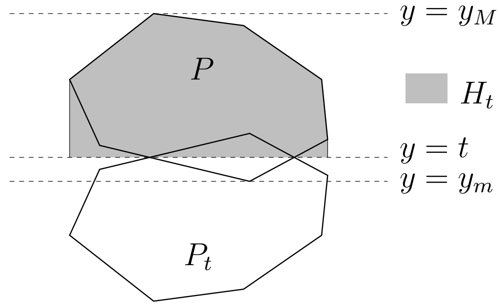

Figure 2: proof of Lemma 15

for $t \in (-\infty, \infty)$. Since exactly half of the area is above the line $y = t$, this reduces to minimizing $|H_t|$. We also note that the minimum necessarily lies in the interval $[y_m, y_M]$ where y_m is the minimum y-coordinate of a vertex of P, and y_M is the maximum.

We conceptually sweep the line $y = t$ from $y = y_m$ to $y = y_M$. When we perform this sweep, the vertices of H_t are either vertices of P (which do not move), or vertices of P_t that move upward at speed 2 (twice the speed of the sweep line $y = t$), or the two vertices with y-coordinate t that move upwards at speed 1. The sequence of vertices along the boundary of H_t changes during the sweep, we call each such change an *event*. There are two kinds of events: either a vertex of P_t hits the boundary of H_t and becomes a new boundary vertex (see Fig. 3), or a vertex is left behind by its two neighbors and disappears from the boundary of H_t. Let $y_m = t_0 < t_1 < t_2 \dots t_k = y_M$

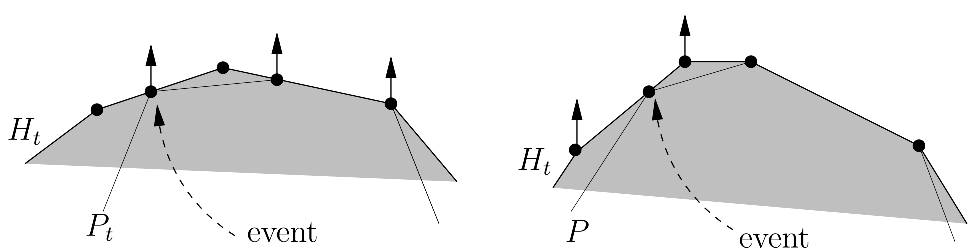

Figure 3: possible events

be the sequence of the times at which events occur during the plane sweep. Since vertices move upward at speed 0, 1 or 2, the function $|H_t|$ is a linear function of t in the interval $t \in [t_i, t_{i+1}]$. So $|H_t|$ achieves its minimum at an event point. We denote by α_i and β_i the numbers such that $|H_t| = \alpha_i t + \beta_i$ in the interval $[t_i, t_{i+1}]$.

The *upper hull* of a convex polygon is the part of its boundary that is locally on or above it. We start by constructing two arrays, containing the vertices of the upper hull of P and P_t, sorted from left to right. The upper hull of P_t is, of course, the lower hull of P.

During the plane sweep, we maintain the sequence of edges of the upper hulls of H_t in a linked list, sorted from left to right. For each edge e of H_t that is not an edge of P_t, we store the index $l(e)$ and $r(e)$ of the leftmost and rightmost vertex of the upper hull of P_t that lies vertically below e.

We keep a set of events in a priority queue that is ordered chronologically. In particular, for each vertex v of H_t that does not move (in other words, this vertex is also a vertex of P) and has at least one moving neighbor, we keep the event where it meets the line segment connecting its two neighbors (in other words, the event where v leaves the boundary of H_t). For each edge e of H_t that is not an edge of P_t, we keep the earliest event where a vertex of the convex chain between $l(e)$ and $r(e)$ hits e. The time and the vertex involved in this event can be found in $O(\log n)$ time by binary search; this still works when e has a vertex moving upward. Finally, we maintain the current value of (α_i, β_i) during the plane sweep.

We process the events in chronological order by repeatedly extracting the next event from the priority queue. Since we know the value of (α_i, β_i) at any event time t_i, we can compute $|H_{t_i}|$ for all i, and obtain the minimum of $|H_t|$ at the end of the sweep. Note that each vertex of P_t can enter the boundary of H_t at most once, and each vertex of P can leave the boundary at most once. It follows that the number of events that occur during the course of this algorithm is $O(n)$. Our data structure requires $O(n)$ space and can be updated in $O(\log n)$ time per event. It can be initialized at time $t = y_m$ in time $O(n)$. Therefore, the overall running time of our algorithm is $O(n \log n)$, which completes the proof of Lemma 15.

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